MA Advanced Macroeconomics:
6. Solving Models with Rational Expectations

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Moving Beyond VARs

- Having described econometric methods for measuring the shocks that hit the macroeconomy and their dynamic effects, we now turn to developing theoretical models that can explain these patterns.
- This requires models with explicit dynamics and with stochastic shocks.
- Obviously, VARs are dynamic stochastic models. VARs, however, are *econometric* not theoretical models and they have their limitations.
- They do not explictly characterise the underlying decision rules adopted by firms and households: They don't tell us *why* things happen.
- This "why" element is crucial if the stories underlying our forecasts or analysis of policy effects are to be believed.
- The goal of the modern DSGE approach is to develop models that can explain macroeconomic dynamics as well as the VAR approach, but that are based upon the fundamental idea of optimising firms and households.

Part I

Introduction to Rational Expectations

Introducing Expectations

- A key sense in which DSGE models differ from VARs is that while VARs just have backward-looking dynamics, DSGE models backward-loking *and* forward-looking dynamics.
- The backward-looking dynamics stem, for instance, from identities linking today's capital stock with last period's capital stock and this period's investment, i.e. $K_t = (1 - \delta)K_{t-1} + I_t$.
- The forward-looking dynamics stem from optimising behaviour: What agents expect to happen tomorrow is very important for what they decide to do today.
- Modelling this idea requires an assumption about how people formulate expectations.
- The DSGE approach relies on the idea that people have so-called rational expectations.
- I will first introduce the idea of rational expectations and describe how to solve and simulate linear rational expecations models that have both backward and forward-looking components.

Rational Expectations

- Almost all economic transactions rely crucially on the fact that the economy is not a "one-period game." Economic decisions have an *intertemporal* element to them.
- A key issue in macroeconomics is how people formulate expectations about the in the presence of uncertainty.
- Prior to the 1970s, this aspect of macro theory was largely *ad hoc*. Generally, it was assumed that agents used some simple extrapolative rule whereby the expected future value of a variable was close to some weighted average of its recent past values.
- This approach criticised in the 1970s by economists such as Robert Lucas and Thomas Sargent. Lucas and Sargent instead promoted the use of an alternative approach which they called "rational expectations."
- In economics, rational expectations usually means two things:
  1. They use publicly available information in an efficient manner. Thus, they do not make systematic mistakes when formulating expectations.
  2. They understand the structure of the model economy and base their expectations of variables on this knowledge.

Rational Expectations as a Baseline

- Rational expectations is clearly a strong assumption.
- The structure of the economy is complex and in truth nobody truly knows how everything works.
- But one reason for using rational expectations as a baseline assumption is that once one has specified a particular model of the economy, any other assumption about expectations means that people are making systematic errors, which seems inconsistent with rationality.
- Still, behavioural economists have now found lots of examples of deviations from rationality in people's economic behaviour.
- But rational expectations requires one to be explicit about the full limitations of people's knowledge and exactly what kind of mistakes they make. And while rational expectations is a clear baseline, once one moves away from it there are lots of essentially *ad hoc* potential alternatives.
- At least at present, the profession has no clear agreed alternative to rational expectations as a baseline assumption.
- And like all models, rational expectations models need to be assessed on the basis of their ability to fit the data.

Part II

Single Stochastic Difference Equations

First-Order Stochastic Difference Equations

- Lots of models in economics take the form

$$y_t = x_t + aE_t y_{t+1}$$

- The equation just says that y today is determined by x and by tomorrow's expected value of y. But what determines this expected value? Rational expectations implies a very specific answer.
- Under rational expectations, the agents in the economy understand the equation and formulate their expectation in a way that is consistent with it:

$$E_t y_{t+1} = E_t x_{t+1} + aE_t E_{t+1} y_{t+2}$$

  This last term can be simplified to

$$E_t y_{t+1} = E_t x_{t+1} + aE_t y_{t+2}$$

  because $E_t E_{t+1} y_{t+2} = E_t y_{t+2}$.
- This is known as the *Law of Iterated Expectations*: It is not rational for me to expect to have a different expectation next period for y_{t+2} than the one that I have today.

Repeated Substitution

- Substituting this into the previous equation, we get

$$y_t = x_t + aE_t x_{t+1} + a^2 E_t y_{t+2}$$

- Repeating this by substituting for $E_t y_{t+2}$, and then $E_t y_{t+3}$ and so on gives

$$y_t = x_t + aE_t x_{t+1} + a^2 E_t x_{t+2} + ... + a^{N-1} E_t x_{t+N-1} + a^N E_t y_{t+N}$$

- Which can be written in more compact form as

$$y_t = \sum_{k=0}^{N-1} a^k E_t x_{t+k} + a^N E_t y_{t+N}$$

- Usually, it is assumed that

$$\lim_{N \to \infty} a^N E_t y_{t+N} = 0$$

- So the solution is

$$y_t = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$

This solution underlies the logic of a very large amount of modern macroeconomics.

Example: Asset Pricing

- Consider an asset that can be purchased today for price P_t and which yields a dividend of D_t. Suppose there is a close alternative to this asset that will yield a guaranteed rate of return of r.
- Then, for a risk neutral investor will only hold the asset if it yields the same rate of return, i.e. if
$$\frac{D_t + E_t P_{t+1}}{P_t} = 1 + r$$
- This can be re-arranged to give
$$P_t = \frac{D_t}{1+r} + \frac{E_t P_{t+1}}{1+r}$$
- The repeated substitution solution is
$$P_t = \sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^{k+1} E_t D_{t+k}$$
- This equation, which states that asset prices should equal a discounted present-value sum of expected future dividends, is usually known as the *dividend-discount model*.

"Backward" Solutions

- The model

$$y_t = x_t + aE_t y_{t+1}$$

can also be written as

$$y_t = x_t + ay_{t+1} + a\epsilon_{t+1}$$

where ϵ_{t+1} is a forecast error that cannot be predicted at date t.

- Moving the time subscripts back one period and re-arranging this becomes

$$y_t = a^{-1}y_{t-1} - a^{-1}x_{t-1} - \epsilon_t$$

- This backward-looking equation which can also be solved via repeated substitution to give

$$y_t = -\sum_{k=0}^{\infty} a^{-k}\epsilon_{t-k} - \sum_{k=1}^{\infty} a^{-k}x_{t-k}$$

Choosing Between Forward and Backward Solutions

- The forward and backward solutions are both correct solutions to the first-order stochastic difference equation (as are all linear combinations of them). Which solution we choose to work with depends on the value of the parameter a.
- If $|a| > 1$, then the weights on future values of x_t in the forward solution will explode. In this case, it is most likely that the forward solution will not converge to a finite sum. Even if it does, the idea that today's value of y_t depends more on values of x_t far in the distant future than it does on today's values is not one that we would be comfortable with. In this case, practical applications should focus on the backwards solution.
- However, the equation holds for any set of shocks ϵ_t such that $E_{t-1}\epsilon_t = 0$. So the solution is *indeterminate*: We can't actually predict what will happen with y_t even if we know the full path for x_t.
- But if $|a| < 1$ then the weights in the backwards solution are explosive and the forward solution is the one to focus on. Also, this solution is determinate. Knowing the path of x_t will tell you the path of y_t.

Rational Bubbles

- In most cases, it is assumed that $|a| < 1$.
- In this case, the assumption that

$$\lim_{N\to\infty} a^N E_t y_{t+N} = 0$$

amounts to a statement that y_t can't grow too fast.

- What if it doesn't hold? Then the solution can have other elements.
- Let

$$y_t^* = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$

- And let $y_t = y_t^* + b_t$ be any other solution. The solution must satisfy

$$y_t^* + b_t = x_t + aE_t y_{t+1}^* + aE_t b_{t+1}$$

- By construction, one can show that $y_t^* = x_t + aE_t y_{t+1}^*$.

Rational Bubbles, Continued

- This means the additional component satisfies

$$b_t = aE_t b_{t+1}$$

- Because $|a| < 1$, this means b is always expected to get bigger in absolute value, going to infinity in expectation. This is a *bubble*.
- Note that the term bubbles is usually associated with irrational behaviour by investors. But, in this model, the agents have rational expectations. This is a rational bubble.
- There may be restrictions in the real economy that stop b growing forever. But constant growth is not the only way to satisfy $b_t = aE_t b_{t+1}$. The following process also works:

$$b_{t+1} = \begin{cases} (aq)^{-1} b_t + e_{t+1} & \text{with probability} \quad q \\ e_{t+1} & \text{with probability} \quad 1-q \end{cases}$$

where $E_t e_{t+1} = 0$.
- This is a bubble that everyone knows is going to crash eventually. And even then, a new bubble can get going. Imposing $\lim_{N\to\infty} a^N E_t y_{t+N} = 0$ rules out bubbles of this (or any other) form.

From Structural to Reduced Form Relationships

- The solution
$$y_t = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$
provides useful insights into how the variable y_t is determined.
- However, without some assumptions about how x_t evolves over time, it cannot be used to give precise predictions about they dynamics of y_t.
- Ideally, we want to be able to simulate the behaviour of y_t on the computer.
- One reason there is a strong linkage between DSGE modelling and VARs is that this question is usually addressed by assuming that the exogenous "driving variables" such as x_t are generated by backward-looking time series models like VARs.
- Consider for instance the case where the process driving x_t is
$$x_t = \rho x_{t-1} + \epsilon_t$$
where $|\rho| < 1$.

From Structural to Reduced Form Relationships, Continued

- In this case, we have
$$E_t x_{t+k} = \rho^k x_t$$
- Now the model's solution can be written as
$$y_t = \left[\sum_{k=0}^{\infty} (a\rho)^k \right] x_t$$
- Because $|a\rho| < 1$, the infinite sum converges to
$$\sum_{k=0}^{\infty} (a\rho)^k = \frac{1}{1 - a\rho}$$
Remember this identity from the famous Keynesian multiplier formula.
- So, in this case, the model solution is
$$y_t = \frac{1}{1 - a\rho} x_t$$
- Macroeconomists call this a *reduced-form* solution for the model: Together with the equation descrining the evolution of x_t, it can easily be simulated on a computer.

The DSGE Recipe

- While this example is obviously a relatively simple one, it illustrates the general principal for getting predictions from DSGE models:
    1. Obtain *structural* equations involving expectations of future driving variables, (in this case the $E_t x_{t+k}$ terms).
    2. Make assumptions about the time series process for the *driving variables* (in this case x_t)
    3. Solve for a *reduced-form* solution than can be simulated on the computer along with the driving variables.
- Finally, note that the reduced-form of this model also has a VAR-like representation, which can be shown as follows:

$$\begin{aligned} y_t &= \frac{1}{1-a\rho}\left(\rho x_{t-1}+\epsilon_t\right) \\ &= \rho y_{t-1}+\frac{1}{1-a\rho}\epsilon_t \end{aligned}$$

So both the x_t and y_t series have purely backward-looking representations. Even this simple model helps to explain how theoretical models tend to predict that the data can be described well using a VAR.

Another Example: The Permanent Income Hypothesis

- Consider, for example, a simple "permanent income" model in which consumption depends on a present discounted value of after-tax income

$$c_t = \gamma \sum_{k=0}^{\infty} \beta^k E_t y_{t+k}$$

- Suppose that income has followed the process

$$y_t = (1+g)\, y_{t-1} + \epsilon_t$$

- In this case, we have

$$E_t y_{t+k} = (1+g)^k\, y_t$$

- So the reduced-form representation is

$$c_t = \gamma \left[\sum_{k=0}^{\infty} \left(\beta\,(1+g)\right)^k \right] y_t$$

  Assuming that $\beta\,(1+g) < 1$, this becomes

$$c_t = \frac{\gamma}{1-\beta\,(1+g)}\, y_t$$

The Lucas Critique

- Think about this example. The structural equation

$$c_t = \gamma \sum_{k=0}^{\infty} \beta^k E_t y_{t+k}$$

  is *always* true for this model

- But the reduced-form representation

$$c_t = \frac{\gamma}{1 - \beta(1+g)} y_t$$

  depends on the process for y_t taking a particular form. Should that process change, the reduced-form process will change.

- In a famous 1976 paper, Robert Lucas pointed out that the assumption of rational expectations implied that the coefficients in reduced-form models would change if expectations about the future changed.

- Lucas stressed that this could make reduced-form econometric models based on historical data useless for policy analysis. This problem is now known as the *Lucas critique* of econometric models.

An Example: Temporary Tax Cuts

- Suppose the government is thinking about a temporary one-period income tax cut. Consider y_t to be after-tax labour income, so it would be temporarily boosted by the tax cut.
- They ask their economic advisers for an estimate of the effect on consumption of the tax cut. The advisers run a regression of consumption on after-tax income.
- If, in the past, consumers had generally expected income growth of g, then these regressions will produce a coefficient of approximately $\frac{\gamma}{1-\beta(1+g)}$ on income. So, the advisers conclude that for each €1 of income produced by the tax cut, there will be an increase in consumption of € $\frac{\gamma}{1-\beta(1+g)}$.
- But if the households have rational expectations, then then each €1 of tax cut will produce only €γ of extra consumption.
- Suppose $\beta = 0.95$ and $g = 0.02$. In this case, the advisor concludes that each unit of tax cuts is worth extra 32γ $\left(=\frac{\gamma}{1-\beta(1+g)}\right)$ in consumption. In reality, the tax cut will produce only γ units of extra consumption. Being off by a factor of 32 constitutes a big mistake in assessing the effect of this policy.

The Lucas Critique and the Limitations of VAR Analysis

- The tax cut example gets the logic of the critique across but perhaps not its generality.
- Today's DSGE models feature policy equations that describe how monetary policy is set via rules relating interest rates to inflation and unemployment; how fiscal variables depends on other macro variables; what the exchange rate regime is.
- These models all feature rational expectations, so changes to these policy rules will be expected to alter the reduced-form VAR-like structures associated with these economies.
- This is an important "selling point" for modern DSGE models. These models can explain why VARs fit the data well, but they can be considered superior tools for policy analysis.
- They explain how reduced-form VAR-like equations are generated by the processes underlying policy and other driving variables. However, while VAR models do not allow reduced-form correlations change over time, a fully specified DSGE model can explain such patterns as the result of structural changes in policy rules.

Second-Order Stochastic Difference Equations

- Variables that are characterized by

$$y_t = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$

  are *jump variables*. They only depends on what's happening today and what's expected to happen tomorrow. If expectations about the future change, they will jump. Nothing that happened in the past will restrict their movement.

- This may be an ok characterization of financial variables like stock prices but it's harder to argue with it as a description of variables in the real economy like employment, consumption or investment.

- Many models in macroeconomics feature variables which depend on *both* the expected future values and their past values. They are characterized by second-order difference equations of the form

$$y_t = a y_{t-1} + b E_t y_{t+1} + x_t$$

Solving Second-Order Stochastic Difference Equations

- Here's one way of solving second-order SDEs. Suppose there was a value λ such that

$$v_t = y_t - \lambda y_{t-1}$$

followed a first-order stochastic difference equation of the form

$$v_t = \alpha E_t v_{t+1} + \beta x_t$$

We'd know how to solve that for v_t and then back out the values for y_t.

- From the fact that $y_t = v_t + \lambda y_{t-1}$, we can re-write the original equation as

$$\begin{aligned} v_t + \lambda y_{t-1} &= a y_{t-1} + b \left(E_t v_{t+1} + \lambda y_t \right) + x_t \\ &= a y_{t-1} + b E_t v_{t+1} + b\lambda \left(v_t + \lambda y_{t-1} \right) + x_t \end{aligned}$$

- This re-arranges to give

$$(1 - b\lambda) v_t = b E_t v_{t+1} + x_t + \left(b\lambda^2 - \lambda + a \right) y_{t-1}$$

Solving Second-Order Stochastic Difference Equations

- By definition, λ was a number such that the v_t it defined followed a first-order stochastic difference equation. This means that λ satisfies:

$$b\lambda^2 - \lambda + a = 0$$

- This is a quadratic equation, so there are two values of λ that satisfy it. For either of these values, we can characterize v_t by

$$\begin{aligned} v_t &= \frac{b}{1-b\lambda} E_t v_{t+1} + \frac{1}{1-b\lambda} x_t \\ &= \frac{1}{1-b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1-b\lambda} \right)^k E_t x_{t+k} \end{aligned}$$

- And y_t obeys

$$y_t = \lambda y_{t-1} + \frac{1}{1-b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1-b\lambda} \right)^k E_t x_{t+k}$$

- Usually, only one of the potential values of λ is less than one in absolute value, so this delivers the unique stable solution.

Example: A Hybrid New Keynesian Phillips Curve

- Last term, we introduced the so-called New Keynesian Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + \nu x_t,$$

where x_t is a measure of inflationary pressures.

- Many empirical studies have suggested that this formulation has difficulty in explaining the persistence observed in the inflation data.

- Some have proposed a "hybrid" variant:

$$\pi_t = \gamma_f E_t \pi_{t+1} + \gamma_b \pi_{t-1} + \kappa x_t$$

with the lagged element coming from some fraction of the population being non-rational price-setters who rely on past inflation for their current behaviour.

- The solution for this model takes the form

$$\pi_t = \lambda \pi_{t-1} + \frac{\kappa}{1 - \gamma_f \lambda} \sum_{k=0}^{\infty} \left(\frac{\gamma_f}{1 - \gamma_f \lambda} \right)^k E_t x_{t+k}$$

- where λ is a solution to

$$\gamma_f \lambda^2 - \lambda + \gamma_b = 0$$

Example: A Hybrid New Keynesian Phillips Curve

- In general, there will be two possible values of λ to solve the so-called characteristic equation of the model. Usually, only one of these values will work as the λ in this formulation.
- Consider the case where the model is

$$\pi_t = \theta E_t \pi_{t+1} + (1-\theta)\pi_{t-1} + \kappa x_t$$

  In this case, the possible solutions of the characteristic equation are $\lambda_1 = 1$ and $\lambda_2 = \frac{1-\theta}{\theta}$.
- If $0 < \theta \leq 0.5$, then the stable solution is

$$\pi_t = \pi_{t-1} + \frac{\kappa}{1-\theta} \sum_{k=0}^{\infty} \left(\frac{\theta}{1-\theta} \right)^k E_t x_{t+k}$$

- Alternatively if $0.5 \leq \theta < 1$, then the stable solution is

$$\pi_t = \left(\frac{1-\theta}{\theta} \right) \pi_{t-1} + \frac{\kappa}{\theta} \sum_{k=0}^{\infty} E_t x_{t+k}$$

Part III

Systems of Stochastic Difference Equations

Systems of Rational Expectations Equations

- So far, we have only looked at a single equation linking two variables. However, it turns out that the logic of the first-order stochastic difference equation underlies the solution methodology for just about all rational expectations models.
- Suppose one has a vector of variables

$$Z_t = \begin{pmatrix} z_{1t} \\ z_{2t} \\ . \\ z_{nt} \end{pmatrix}$$

- It turns out that a lot of macroeconomic models can be represented by an equation of the form

$$Z_t = BE_t Z_{t+1} + X_t$$

where B is an $n \times n$ matrix. The logic of repeated substitution can also be applied to this model, to give a solution of the form

$$Z_t = \sum_{k=0}^{\infty} B^k E_t X_{t+k}$$

Eigenvalues

- As with the single-equation model, this will only give a stable non-explosive solution under certain conditions.
- A value λ_i is an eigenvalue of the matrix B if there exists a vector e_i (known as an eigenvector) such that
$$Be_i = \lambda_i e_i$$
- Many $n \times n$ matrices have n distinct eigenvalues. Denote by P the matrix that has as its columns n eigenvectors corresponding to these eigenvalues. In this case,
$$BP = P\Omega$$
where
$$\Omega = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$$
is a diagonal matrix of eigenvalues.

Stability Condition

- Note now that this equation implies that

$$B = P \Omega P^{-1}$$

- This tells us something about the relationship between eigenvalues and higher powers of B because

$$B^n = P \Omega^n P^{-1} = P \begin{pmatrix} \lambda_1^n & 0 & 0 & 0 \\ 0 & \lambda_2^n & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & \lambda_n^n \end{pmatrix} P^{-1}$$

- So, the difference between lower and higher powers of B is that the higher powers depend on the eigenvalues taken to the power of n. If all of the eigenvalues are inside the unit circle (i.e. less than one in absolute value) then all of the entries in B^n will tend towards zero as $n \to \infty$.
- So, a condition that ensures that a model of the form $Z_t = BE_t Z_{t+1} + X_t$ has a unique stable forward-looking solution is that the eigenvalues of B are all inside the unit circle.

How Are Eigenvalues Calculated

- Consider, for example, a 2×2 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Suppose A has two eigenvalues, λ_1 and λ_2 and define λ as the vector

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

- The fact that there are eigenvectors which when multiplied by $A - \lambda I$ equal a vector of zeros means that the determinant of the matrix

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix}$$

equals zero.

- So solving the quadratic formula

$$(a_{11} - \lambda_1 a_{12})(a_{22} - \lambda_2) - a_{12}a_{21} = 0$$

gives the two eigenvalues of A.

More General Models: The Binder-Pesaran Method

- Consider a vector Z_t characterized by

$$Z_t = AZ_{t-1} + BE_t Z_{t+1} + HX_t$$

- The restriction to one-lag one-lead form is only apparent, and the companion matrix trick can be used to allow this model to represent models with n leads and lags. In this sense, this equation summarizes all possible linear rational expectations models.
- Binder and Pesaran (1996) solved this model in a manner exactly analagous to the second-order difference equation discussed earlier. Find a matrix C such that $W_t = Z_t - CZ_{t-1}$ obeys a first-order matrix equation of the form

$$W_t = FE_t W_{t+1} + GX_t$$

- In other words, we transform the problem of solving the "second-order" system in equation into a simpler first-order system.

More General Models: The Binder-Pesaran Method

- What must the matrix C be? Using the fact that

$$Z_t = W_t + CZ_{t-1}$$

- The model can be re-written as

$$\begin{aligned} W_t + CZ_{t-1} &= AZ_{t-1} + B\left(E_t W_{t+1} + CZ_t\right) + HX_t \\ &= AZ_{t-1} + B\left(E_t W_{t+1} + C\left(W_t + CZ_{t-1}\right)\right) + HX_t \end{aligned}$$

- This re-arranges to

$$\left(I - BC\right) W_t = BE_t W_{t+1} + \left(BC^2 - C + A\right) Z_{t-1} + HX_t$$

- Because C is the matrix that such that W_t follows a first-order forward-looking matrix equation (with no extra Z_{t-1} terms) it follows that

$$BC^2 - C + A = 0$$

- This "matrix quadratic equation" can be solved to give C. Solving these equations is non-trivial (see paper on the website). One method uses the fact that $C = BC^2 + A$, to solve for it iteratively as follows. Provide an initial guess, say $C_0 = I$, and then iterate on $C_n = BC_{n-1}^2 + A$ until all the entries in C_n converge.

Model Solution

- Once we know C, we have

$$W_t = FE_t W_{t+1} + GX_t$$

where

$$\begin{aligned} F &= (I - BC)^{-1} B \\ G &= (I - BC)^{-1} H \end{aligned}$$

- Assuming the all the eigenvalues of F are inside the unit circle, this has a stable forward-looking solution

$$W_t = \sum_{k=0}^{\infty} F^k E_t \left(GX_{t+k} \right)$$

which can be written in terms of the original equation as

$$Z_t = CZ_{t-1} + \sum_{k=0}^{\infty} F^k E_t \left(GX_{t+k} \right)$$

Reduced-Form Representation

- Suppose we assume that the driving variables X_t follow a VAR representation of the form

$$X_t = DX_{t-1} + \epsilon_t$$

  where D has eigenvalues inside the unit circle.

- This implies $E_t X_{t+k} = D^k X_t$, so the model solution is

$$Z_t = CZ_{t-1} + \left[\sum_{k=0}^{\infty} F^k G D^k \right] X_t$$

- The infinite sum in this equation will converge to a matrix P, so the model has a reduced-form representation

$$Z_t = CZ_{t-1} + PX_t$$

  which can be simulated along with the VAR process for the driving variables.

- This provides a relatively simple recipe for simulating DSGE models: Specify the A, B and H matrices; solve for C, F and G; specify a VAR process for the driving variables; and then obtain the reduced-form representations.

General Formulation

- The equations we get from models will often contain multiple values of different variables at time t.
- This isn't a problem. We can plug the model into a computer program as

$$KZ_t = AZ_{t-1} + BE_t Z_{t+1} + HX_t$$

- Then the program can multiply both sides by K^{-1} to give

$$Z_t = K^{-1}AZ_{t-1} + K^{-1}BE_t Z_{t+1} + K^{-1}HX_t$$

- Which is a format that can be solved using the Binder-Pesaran method.
- All of this seems a bit complicated. In practice, it's not so hard. You figure out what your model implies in terms of the K, A, B and H matrices (most of the entries are usually zero). Then the computer gives you representation of the form

$$\begin{aligned} Z_t &= CZ_{t-1} + PX_t \\ X_t &= DX_{t-1} + \epsilon_t \end{aligned}$$

which you can start to do calculations with.