MA Macroeconomics
11. The Solow Model

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The Solow Model

- Recall that economic growth can come from capital deepening or from improvements in total factor productivity.
- Implies growth can come about from saving and investment or from improvements in productive efficiency.
- This lecture looks at a model examining role these two elements play in achieving sustained economic growth.
- The model was developed by Robert Solow, whose work on growth accounting we discussed in the last lecture.

Production Function

- Assume a production function in which output depends upon capital and labour inputs as well as a technological efficiency parameter, A.

$$Y_t = AF\left(K_t, L_t\right)$$

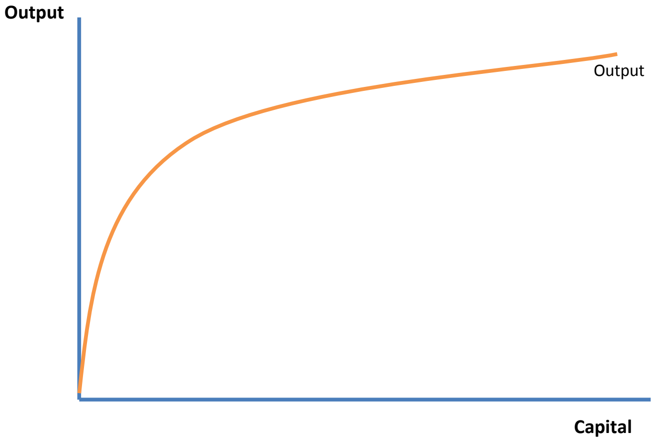
- It is assumed that adding capital and labour raises output

$$\begin{aligned}\frac{\partial Y_t}{\partial K_t} &> 0 \\ \frac{\partial Y_t}{\partial L_t} &> 0\end{aligned}$$

- However, there are diminishing marginal returns to capital accumulation, so extra amounts of capital gives progressively smaller and smaller increases in output.
- This means the second derivative of output with respect to capital is negative.

$$\frac{\partial^2 Y_t}{\partial K_t} < 0$$

Diminishing Returns



Further Assumptions

- Closed economy with no government sector or international trade. This means all output takes the form of either consumption or investment

$$Y_t = C_t + I_t$$

- And that savings equals investment

$$S_t = Y_t - C_t = I_t$$

- Stock of capital changes over time according to

$$\frac{dK_t}{dt} = I_t - \delta K_t$$

- Change in capital stock each period depends positively on savings and negatively on depreciation, which is assumed to take place at rate δ.
- Assumes that consumers save a constant fraction s of their income

$$S_t = sY_t$$

Capital Dynamics in the Solow Model

- Because savings equals investment in the Solow model, this means investment is also a constant fraction of output

$$I_t = sY_t$$

- So we can re-state the equation for changes in the stock of capital

$$\frac{dK_t}{dt} = sY_t - \delta K_t$$

- Whether the capital stock expands, contracts or stays the same depends on whether investment is greater than, equal to or less than depreciation.

$$\begin{aligned} \frac{dK_t}{dt} > 0 &\quad \text{if} \quad \delta K_t < sY_t \\ \frac{dK_t}{dt} = 0 &\quad \text{if} \quad \delta K_t = sY_t \\ \frac{dK_t}{dt} < 0 &\quad \text{if} \quad \delta K_t > sY_t \end{aligned}$$

Capital Dynamics

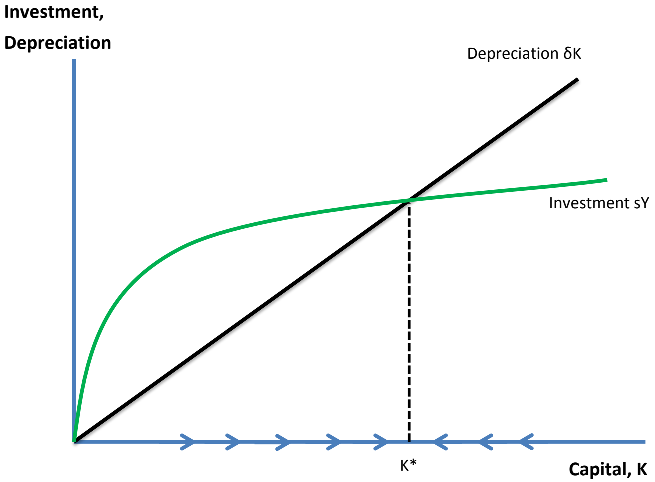
- If the ratio of capital to output is such that

$$\frac{K_t}{Y_t} = \frac{s}{\delta}$$

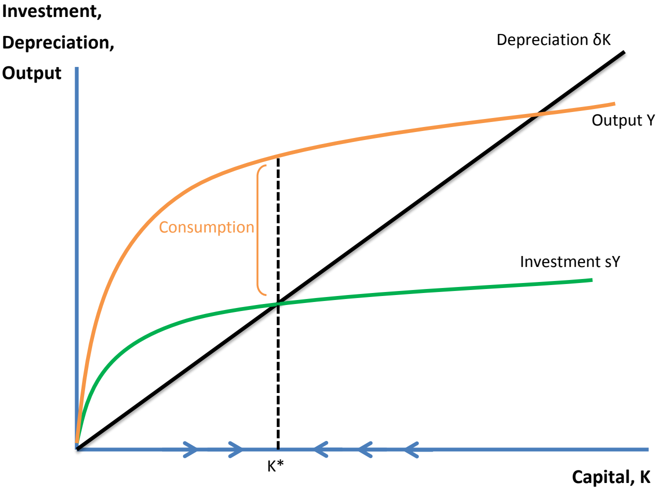
  then the stock of capital will stay constant.

- When the level of capital is low, sY_t is greater than δK. As the capital stock increases, the additional investment tails off but the additional depreciation does not, so at some point sY_t equals δK.
- If we start out with a high stock of capital, then depreciation, δK, will tend to be greater than investment, sY_t and the stock of capital will decline until it reaches K^*.
- This an example of what economists call *convergent dynamics*.
- If nothing else in the model changes, there will be a defined level of capital that the economy converges towards, no matter where the capital stock starts.

Capital Dynamics in The Solow Model



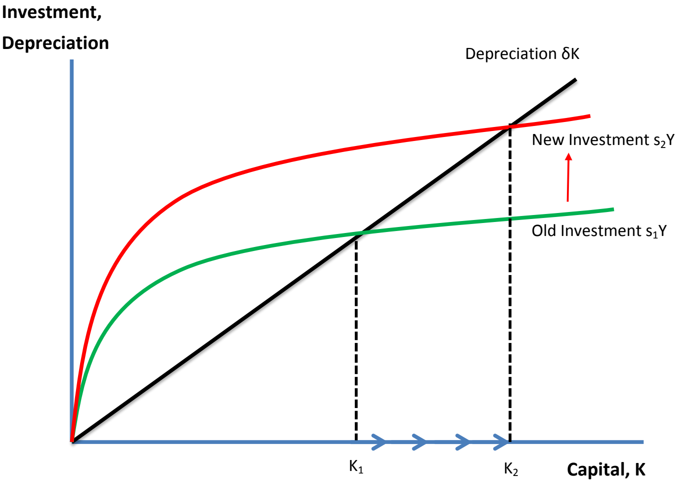
The Solow Model: Capital and Output



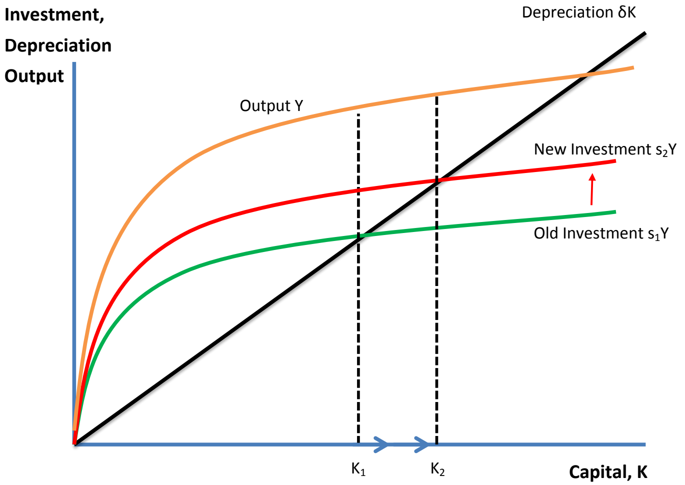
Effect of a Change in Savings

- Now consider what happens when the economy has settled down at an equilibrium unchanging level of capital K_1 and then there is an increase in the savings rate from s_1 to s_2.
- Line for investment shifts upwards: For each level of capital, the level of output associated with it translates into more investment.
- Starting at the initial level of capital, K_1, investment now exceeds depreciation.
- This means the capital stock starts to increase until it reaches its new equilibrium level of K_2.

The Solow Model: Increase in Investment



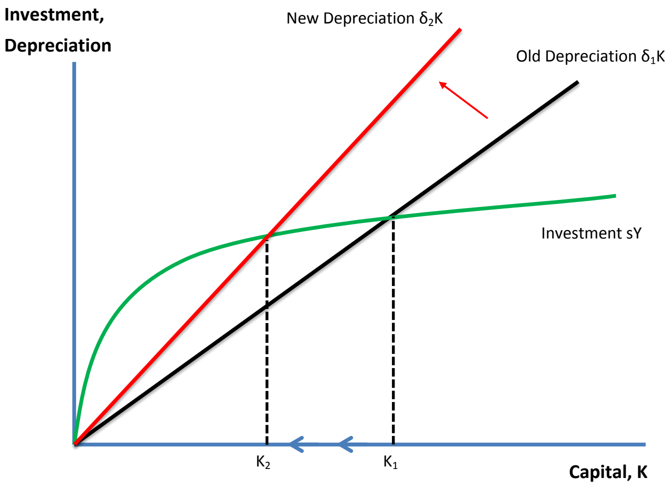
The Solow Model: Effect on Output of Higher Investment



Effect of a Change in Depreciation

- Now consider what happens when the economy has settled down at an equilibrium level of capital K_1 and then there is an increase in the depreciation rate from δ_1 to δ_2.
- The depreciation schedule shifts up from the original depreciation rate, δ_1, to the new schedule associated with δ_2.
- Starting at the initial level of capital, K_1, depreciation now exceeds investment.
- This means the capital stock starts to decline, and continues until capital falls to its new equilibrium level of K_2.
- The increase in the depreciation rate leads to a decline in the capital stock and in the level of output.

The Solow Model: Increase in Depreciation



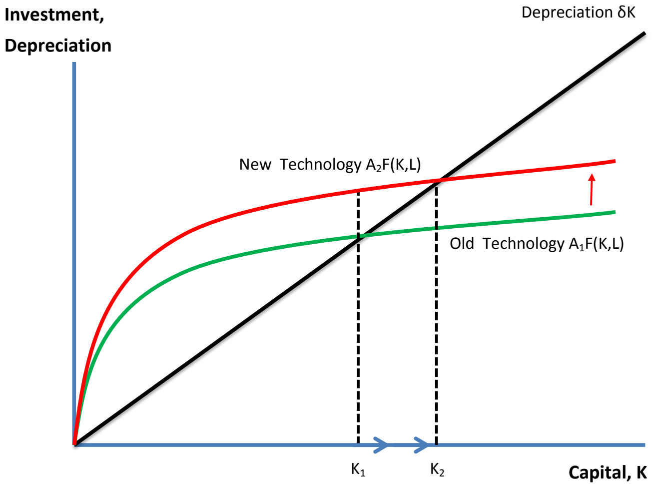
Increase in Technological Efficiency

- Now consider what happens when technological efficiency A_t increases.
- Because investment is given by

$$I_t = sY_t = sAF\left(K_t, L_t\right)$$

- a one-off increase in A thus has the same effect as a one-off increase in s.
- Capital and output gradually rise to a new higher level.

The Solow Model: Increase in Technological Efficiency



Technology Versus Savings as Sources of Growth

- The Solow model shows a one-off increase in technological efficiency, A_t, has same effects as a one-off increase in the savings rate, s.
- However, there are likely to be limits in any economy to the fraction of output that can be allocated towards saving and investment, particularly if it is a capitalist economy in which savings decisions are made by private citizens.
- On the other hand, there is no particular reason to believe that technological efficiency A_t has to have an upper limit. Indeed, growth accounting studies tend to show steady improvements over time in A_t in most countries.
- Going back to Young's paper on Hong Kong and Singapore discussed in the last lecture, you can see now why it matters whether an economy has grown due to capital deepening or TFP growth.
- The Solow model predicts that a policy of encouraging growth through more capital accumulation will tend to tail off over time producing a once-off increase in output per worker. In contrast, a policy that promotes the growth rate of TFP can lead to a sustained higher growth rate of output per worker.

Why Growth Accounting Can Be Misleading

- Consider a country that has a constant share of GDP allocated to investment but is experiencing steady growth in TFP.
- The Solow model predicts that this economy should experience steady increases in output per worker and increases in the capital stock.
- A growth accounting exercise may conclude that a certain percentage of growth stems from capital accumulation.
- But ultimately, in this case, all growth (including the growth in the capital stock) actually stems from growth in TFP.
- The moral here is that pure accounting exercises may miss the ultimate cause of growth.

Krugman on the Soviet Union

- In “The Myth of Asia's Miracle”, Krugman discusses a number of examples of cases where economies where growth was based on largely on capital accumulation. He includes the case of Asian economies like Singapore, which we dicussed previously.
- Another interesting case he focuses on is the economy of the Soviet Union. The Soviet grew strongly after World War 2 and many predicted would overtake Western economies.
- However, some economists that examined the Soviet economy were less impressed (longer quote in notes).

  “*But what they actually found was that Soviet growth was based on rapid–growth in inputs–end of story. The rate of efficiency growth was not only unspectacular, it was well below the rates achieved in Western economies. Indeed, by some estimates, it was virtually nonexistent.... [B]ecause input-driven growth is an inherently limited process, Soviet growth was virtually certain to slow down. Long before the slowing of Soviet growth became obvious, it was predicted on the basis of growth accounting.*”

The Capital-Output Ratio with Steady Growth

- Consider how the capital stock behaves when the economy grows at steady constant rate G^Y.
- The capital output ratio $\frac{K_t}{Y_t}$ can be written as $K_t Y_t^{-1}$. So the growth rate of the capital-output ratio can be written as

$$G_t^{\frac{K}{Y}} = G_t^K - G_t^Y$$

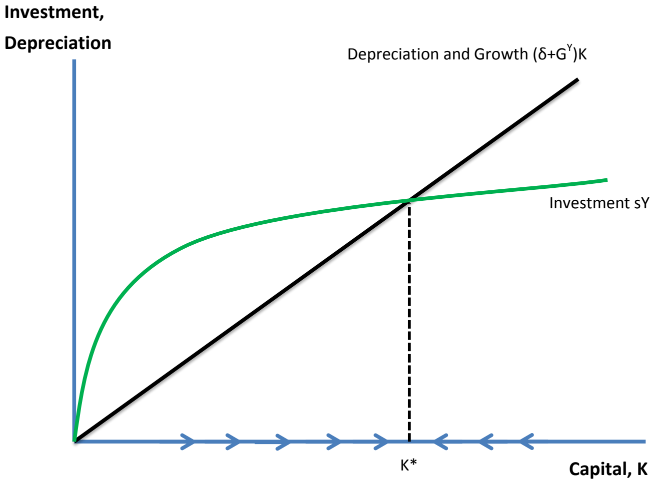
- This means the the growth rate of the capital-output ratio is

$$G_t^{\frac{K}{Y}} = s\frac{Y_t}{K_t} - \delta - G^Y$$

- Convergence dynamics for the capital-output ratio:

$$\begin{aligned} G_t^{\frac{K}{Y}} > 0 \quad & \text{if} \quad \frac{K_t}{Y_t} < \frac{s}{\delta + G^Y} \\ G_t^{\frac{K}{Y}} = 0 \quad & \text{if} \quad \frac{K_t}{Y_t} = \frac{s}{\delta + G^Y} \\ G_t^{\frac{K}{Y}} < 0 \quad & \text{if} \quad \frac{K_t}{Y_t} > \frac{s}{\delta + G^Y} \end{aligned}$$

Capital Dynamics in a Growing Economy



A Formula for Steady Growth

- Cobb-Douglas production function

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

- This means output growth is determined by

$$G_t^Y = G_t^A + \alpha G_t^K + (1-\alpha) G_t^L$$

- Assume $G_t^L = n$ and $G_A^L = g$ then we have

$$G_t^Y = g + \alpha G_t^K + (1-\alpha) n$$

- But we know from capital-output dynamics that capital must be growing at the same rate as output if the growth rate is constant. This gives

$$G_t^Y = \frac{g}{1-\alpha} + n$$

- And the growth rate of output per worker is

$$G_t^Y - n = \frac{g}{1-\alpha}$$

An Alternative Expression for Output Per Worker

- Output per worker with Cobb-Douglas production function is given by

$$\frac{Y_t}{L_t} = A_t \left(\frac{K_t}{L_t} \right)^{\alpha}$$

- But we know increases in A_t also increase capital per worker, so this has misleading implications about the role of capital accumulation.
- An alternative characterisation of output per worker is useful. Define the capital-output ratio as

$$x_t = \frac{K_t}{Y_t}$$

- So, the production function can be expressed as

$$Y_t = A_t \left(x_t Y_t \right)^{\alpha} L_t^{1-\alpha}$$

- Re-arranging this becomes

$$\frac{Y_t}{L_t} = A_t^{\frac{1}{1-\alpha}} x_t^{\frac{\alpha}{1-\alpha}}$$

- This equation states that all fluctuations in output per worker are due to either changes in technological progress or changes in the capital-output ratio.

Some New Terminology

- A useful mathematical shorthand that saves us from having to write down derivatives with respect to time everywhere is to write

$$\dot{Y}_t = \frac{dY_t}{dt}$$

- What we are really interested in, though, is *growth rates* of series, so we need to scale this by the level of output itself.
- Thus, $\frac{\dot{Y}_t}{Y_t}$ is a mathematical expression for the growth rate of a series.

How Does the Capital-Output Ratio Behave?

- Because

$$x_t = K_t Y_t^{-1}$$

its growth rate can be written as

$$\frac{\dot{x}_t}{x_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{Y}_t}{Y_t}$$

- Output growth is

$$\frac{\dot{Y}_t}{Y_t} = g + \alpha \frac{\dot{K}_t}{K_t} + (1-\alpha)n$$

- Capital growth is

$$\frac{\dot{K}_t}{K_t} = s \frac{Y_t}{K_t} - \delta = \frac{s}{x_t} - \delta$$

- So the growth rate of the capital-output ratio is

$$\begin{aligned} \frac{\dot{x}_t}{x_t} &= (1-\alpha)\frac{\dot{K}_t}{K_t} - g - (1-\alpha)n \\ &= (1-\alpha)\left(\frac{s}{x_t} - \frac{g}{1-\alpha} - n - \delta\right) \end{aligned}$$

Convergent Dynamics

- The equation
$$\frac{\dot{x}_t}{x_t} = (1-\alpha)\left(\frac{s}{x_t} - \frac{g}{1-\alpha} - n - \delta\right)$$
has the property that the growth rate of x_t depends negatively on the value of x_t.
- When x_t is over a certain value, it will tend to decline, and when it is under that value it will tend to increase.
- This proves as a general result that, the capital-ouput ratio exhibits convergent dynamics: It tends to converge to a specific long-run steady-state value.
- The equilbrium capital-output ratio is the ratio such that $\frac{\dot{x}_t}{x_t} = 0$. This is
$$x^* = \frac{s}{\frac{g}{1-\alpha} + n + \delta}$$

The Convergence Speed Under Constant Returns

- Multiplying and dividing the previous equation of $\frac{\dot{x}_t}{x_t}$ by $(\frac{g}{1-\alpha} + n + \delta)$:

$$\frac{\dot{x}_t}{x_t} = (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{s/x_t - \frac{g}{1-\alpha} - n - \delta}{\frac{g}{1-\alpha} + n + \delta}\right)$$

- The last term inside the brackets can be simplified to give

$$\begin{aligned}
\frac{\dot{x}_t}{x_t} &= (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{1}{x_t}\frac{s}{\frac{g}{1-\alpha} + n + \delta} - 1\right) \\
&= (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{x^*}{x_t} - 1\right) \\
&= (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{x^* - x_t}{x_t}\right)
\end{aligned}$$

- This equation states that each period the capital-output ratio closes a fraction equal to $\lambda = (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)$ of the gap between the current value of the ratio and its steady-state value.

Illustrating Convergence Dynamics

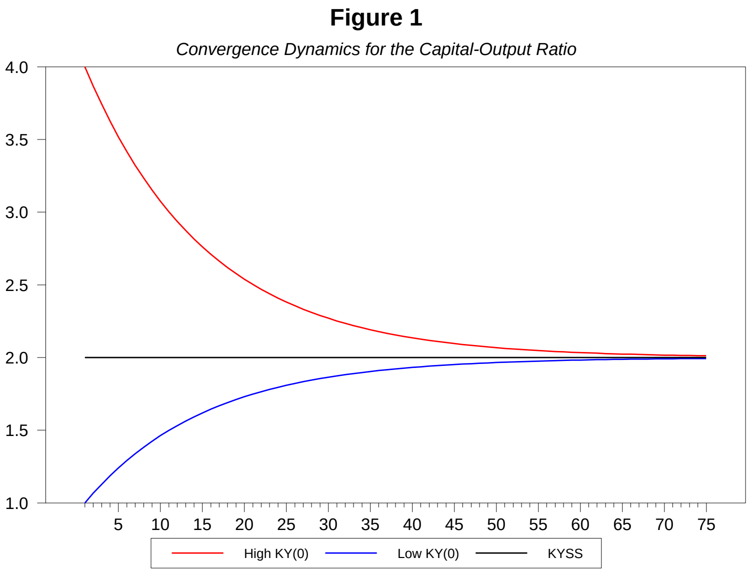
- Figures 1 to 3 provide examples of the behaviour over time of two economies, one that starts with a capital-output ratio that is half the steady-state level, and other that starts with a capital output ratio that is 1.5 times the steady-state level.
- The parameters chosen were $s = 0.2$, $\alpha = \frac{1}{3}$, $g = 0.02$, $n = 0.01$, $\delta = 0.06$. Together these parameters are consistent with a steady-state capital-output ratio of 2.
- To see this, plug these values into the formula for x^*:

$$x^* = \left(\frac{K}{Y}\right)^* = \frac{s}{\frac{g}{1-\alpha} + n + \delta} = \frac{0.2}{1.5 * 0.02 + 0.01 + 0.06} = 2$$

- The first chart shows how the two capital-output ratios converge, somewhat slowly, over time to their steady-state level. This slow convergence is dictated by our choice of parameters: Our "convergence speed" is:

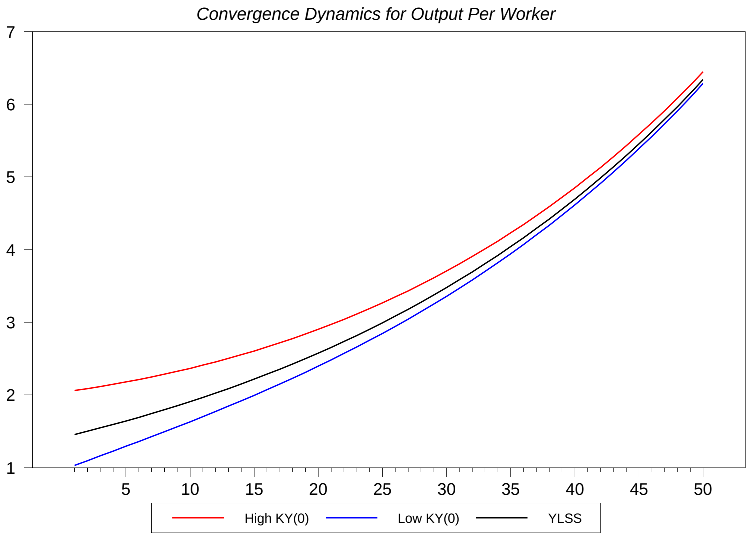
$$\lambda = (1-\alpha)\left(\frac{g}{1-\alpha} + n + \delta\right) = \frac{2}{3}(1.5 * 0.02 + 0.01 + 0.06) = 0.067$$

Convergence Dynamics

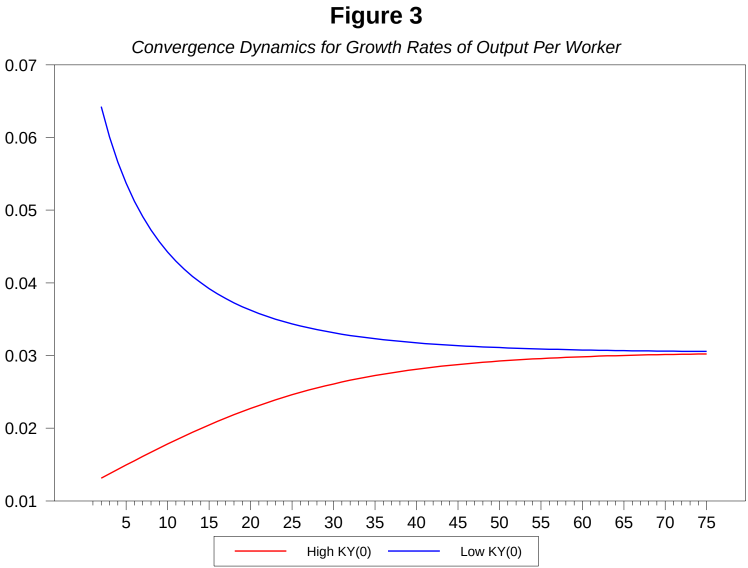


Convergence Dynamics

Figure 2



Convergence Dynamics



Illustrating Changes in Parameters

- Figures 4 to 6 examine what happens when the economy is moving along the steady-state path consistent with the parameters just given, and then one of the parameters is changed. Specifically, it examines the effects of changes in s, δ and g.
- It shows an increase in the savings rate to $s = 0.25$. The growth rate jumps immediately and only slowly returns to the long-run 3 percent value. The faster pace of investment during this period gradually brings the capital-output ratio into line with its new steady-state level.
- The increase in the savings rate permamently raises the level of output per worker relative to the path that would have occurred without the change. However, for our parameter values, this effect is not that big.
- The charts also show the effect of an increase in the depreciation rate to $\delta = 0.11$. This reduces the steady-state capital-output ratio to $4/3$ and the effects of this change are basically the opposite of the effects of the increase in the savings rate.

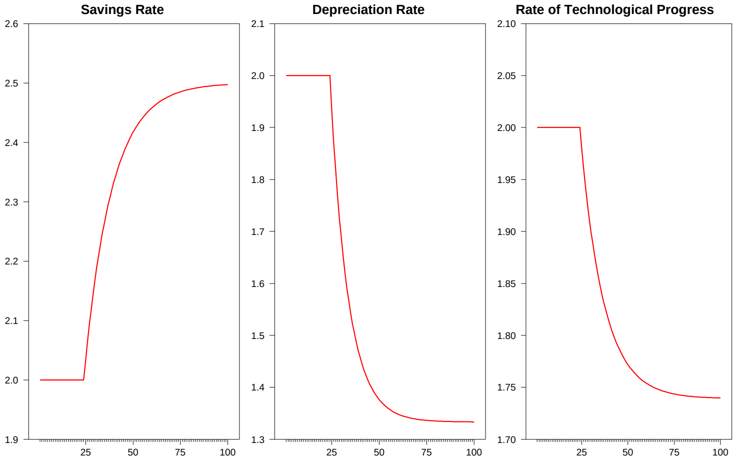
A Change in the Rate of Technological Progress

- Finally, there is the increase in the rate of technological progress. I've shown the effects of a change from $g = 0.02$ to $g = 0.03$.
- This increases the steady-state growth rate of output per worker to 0.045.
- However, as the charts show there is another effect: A faster steady-state growth rate for output reduces the steady-state capital-output ratio. Why? The increase in g raises the long-run growth rate of output; this means that each period the economy needs to accumulate more capital than before just to keep the capital-output ratio constant.
- Without a change in the savings rate that causes this to happen, the capital-output ratio will decline.
- So, the increase in g means that—as in the depreciation rate example—the economy starts out in period 25 with too much capital relative to its new steady-state capital-output ratio. For this reason, the economy doesn't jump straight to its new 4.5 percent growth rate of output per worker. Instead, after an initial jump in the growth rate, there is a very gradual transition the rest of the way to the 4.5 percent growth rate.

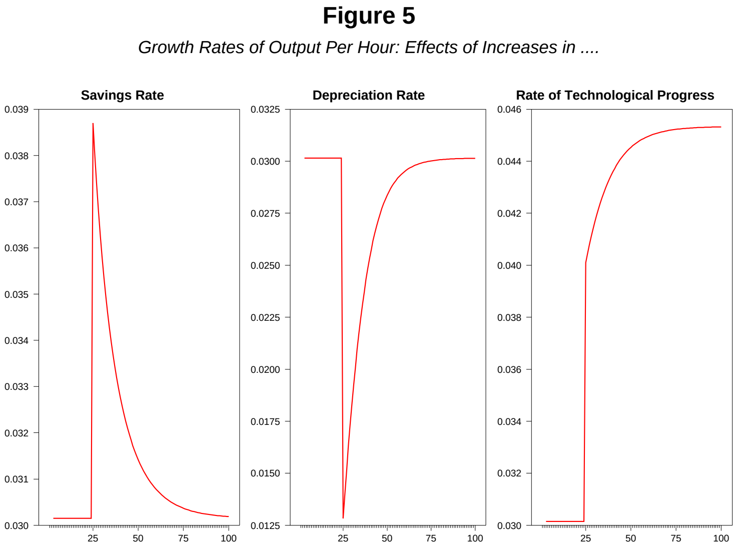
Changing Parameter Values

Figure 4

Capital-Output Ratios: Effects of Increases in ....



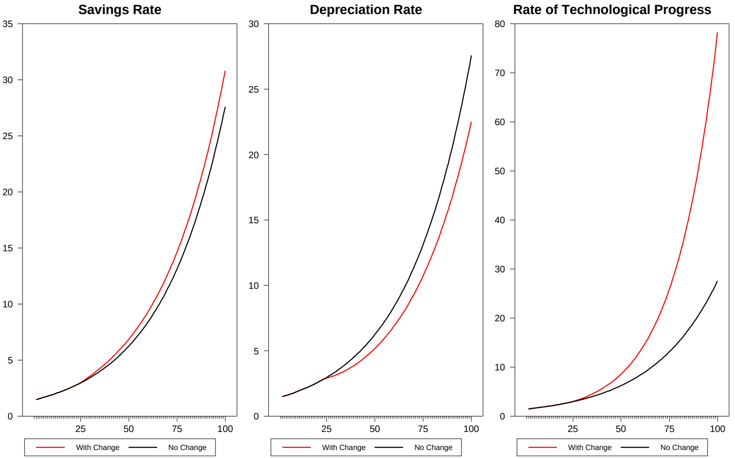
Changing Parameter Values



Changing Parameter Values

Figure 6

Output Per Hour: Effects of Increases in ....



Convergence Dynamics in Practice

- The Solow model predicts economies reach equilibrium levels of output and capital consistent with their underlying features, no matter where they start from.
- Does the evidence support this idea?
- A number of extreme examples show economies having far less capital than is consistent with their fundamental features (e.g. after wars).
- Generally supported Solow's prediction that these economies tend to recover from these setbacks and return to their pre-shock levels of capital and output.
- For example, both Germany and Japan grew very strongly after WW2.
- Another extreme example is study by Edward Miguel and Gerard Roland of the long-run impact of U.S. bombing of Vietnam in the 1960s and 1970s. Despite large differences in the extent of damage inflicted on different regions, Miguel and Roland found little evidence for lasting relative damage on the most-bombed regions by 2002. (Note this is not the same as saying there was no damage to the economy as a whole).

Things to Understand from this Topic

1. The assumptions of the Solow model.
2. The rationale for diminishing marginal returns to capital accumulation.
3. Effects of changes in savings rate, depreciation rate and technology in the Solow model.
4. Why technological progress is the source of most growth.
5. Why growth accounting calculations can underestimate the role of technological progress.
6. Krugman on the Soviet Union.
7. The Solow model's predictions about convergent dynamics.
8. The formula for steady growth rate with a Cobb-Douglas production function.
9. The formula for the convergence rate with a Cobb-Douglas production function.
10. Historical examples of convergent dynamics.