PhD Macroeconomics 1: 9. The Real Business Cycle Model and DSGE Modelling

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The ABCs of RBCs and DSGEs

- Over the past 30 years, macroeconomists have developed a methodology for macroeconomic modelling known as Dynamic Stochastic General Equilibrium (DSGE) modelling.
- The models include optimising firms and households, economy-wide resource constraints and allow for (relatively simplistic) modelling of fiscal and monetary policy.
- We will discuss a well-known example of DSGE model and describe how to use Matlab to solve and simulate these models.
- The methodology for solving and simulating these models has its origins in the methods used to solve the so-called Real Business Cycle (RBC) model. This is a model in which only "'real" shocks affect GDP and monetary policy has no impact.
- It's not a very realistic model but we will describe how to solve it, so you can see how the methods underlying DSGE models evolved.

Part I

Introduction to the Real Business Cycle Model

Working Through A DSGE Model

- We have described methods for solving and simulating linear models with lags, leads and rational expectations.
- Now it is time to go through a particular model to see how these methods get combined with economic theory.
- Specifically, we will work through a version of the Real Business Cycle (RBC) model—introduced in a famous 1982 paper by Finn Kydland and Edward Prescott—is the original DSGE model.[1]
- We will set out a basic RBC model and discuss how the model's first-order conditions can be turned into a system of linear difference equations of the form we know how to solve.
- This will require explaining another new technique, known as *log-linearization*.
- While many now question the specific assumptions underlying the early RBC models, the *methodology* has endured.

[1] "Time to Build and Aggregate Fluctuations," *Econometrica*, November 1982, Volume 50, pages 1345-1370. This paper was cited in the 2004 Nobel prize award given to Kydland and Prescott.

An RBC Model

- The basic RBC model assume perfectly functioning competitive markets, so the outcomes generated by decentralized decisions by firms and households can be replicated as the solution to a social planner problem.
- The social planner wants to maximize

$$E_t \left[\sum_{i=0}^{\infty} \beta^i \left(U(C_{t+i}) - V(N_{t+i}) \right) \right]$$

  where C_t is consumption, N_t is hours worked, and β is the representative household's rate of time preference.
- The economy faces constraints described by

$$\begin{aligned} Y_t &= C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\ K_t &= I_t + (1-\delta) K_{t-1} \end{aligned}$$

  and a process for the technology term A_t, usually a log-linear AR(1):

$$\log A_t = (1-\rho) \log A^* + \rho \log A_{t-1} + \epsilon_t$$

Formulating the Social Planner's Problem

- Remember our two constraints

$$
\begin{aligned}
Y_t &= C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\
K_t &= I_t + (1-\delta) K_{t-1}
\end{aligned}
$$

- We can simplify the problem by combining them into one equation:

$$
A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\delta) K_{t-1}
$$

- We can then formulate the social planner's problem as a Lagrangian problem involving picking a series of values for consumption and labour, subject to satisfying a series of constraints of the form just described:

$$
\begin{aligned}
L &= E_t \sum_{i=0}^{\infty} \beta^i \left[U(C_{t+i}) - V(N_{t+i}) \right] \\
&\quad + E_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} \left[A_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{1-\alpha} + (1-\delta) K_{t+i-1} - C_{t+i} - K_{t+i} \right]
\end{aligned}
$$

How to Get the First-Order Conditions

- This equation might look a bit intimidating. It involves an infinite sum, so technically there is an infinite number of first-order conditions for current and expected future values of C_t, K_t and N_t.

- But the problem is less hard than this makes it sound, Note that the time-t variables appear in this sum as

$$U(C_t) - V(N_t) + \lambda_t \left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\delta) K_{t-1} \right) \\ + \beta E_t \left[\lambda_{t+1} \left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta) K_t \right) \right]$$

- After that, the time-t variables don't ever appear again. So, the FOCs for the time-t variables consist of differentiating this equation with respect to these variables and setting the derivatives equal to zero.

- Then, the time $t+n$ variables appear exactly as the time t variables do, except that they are in expectation form and they are multiplied by the discount rate β^n. But this means the FOCs for the time $t+n$ variables will be identical to those for the time t variables. So differentiating this equation gives us the equations for the optimal dynamics at all times.

The First-Order Conditions

- Differentiating

$$U(C_t) - V(N_t) + \lambda_t \left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\delta) K_{t-1}\right)$$
$$+\beta E_t \left[\lambda_{t+1} \left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta) K_t\right)\right]$$

- We get following first-order conditions:

$$\begin{aligned}
\frac{\partial L}{\partial C_t} &: \quad U'(C_t) - \lambda_t = 0 \\
\frac{\partial L}{\partial K_t} &: \quad -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta\right)\right] = 0 \\
\frac{\partial L}{\partial N_t} &: \quad -V'(N_t) + (1-\alpha) \lambda_t \frac{Y_t}{N_t} = 0 \\
\frac{\partial L}{\partial \lambda_t} &: \quad A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\delta) K_{t-1} = 0
\end{aligned}$$

The Euler Equation (Again)

- Define the marginal value of an additional unit of capital next year as

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_t} + 1 - \delta$$

- Then the FOC for capital can be written as

$$\lambda_t = \beta E_t \left(\lambda_{t+1} R_{t+1} \right)$$

- This can then be combined with the FOC for consumption to give

$$U'(C_t) = \beta E_t \left[U'(C_{t+1}) R_{t+1} \right]$$

- Interpretation:
    - Decrease consumption by Δ today, at a loss of $U'(C_t)\Delta$ in utility.
    - Invest to get $R_{t+1}\Delta$ tomorrow.
    - Worth $\beta E_t \left[U'(C_{t+1}) R_{t+1} \Delta \right]$ in terms of today's utility.
    - Along an optimal path, must be indifferent.

CRRA Consumption and Separable Consumption-Leisure

- We are going to work with a utility function of the form:

$$U(C_t) - V(N_t) = \frac{C_t^{1-\eta}}{1-\eta} - bN_t$$

- This formulation of the Constant Relative Risk Aversion (CRRA) utility from consumption and separate disutility from labour turns out to be necessary for the model to have a stable growth path solution.
- The Euler equation becomes

$$C_t^{-\eta} = \beta E_t \left(C_{t+1}^{-\eta} R_{t+1} \right)$$

- And the condition for optimal hours worked becomes

$$-b + (1-\alpha) C_t^{-\eta} \frac{Y_t}{N_t} = 0$$

The Full Set of Model Equations

- The RBC model can then be defined by the following six equations (three identities describing resource constraints, one a definition, and two FOCs describing optimal behaviour)

$$\begin{aligned}
Y_t &= C_t + I_t \\
Y_t &= A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\
K_t &= I_t + (1-\delta) K_{t-1} \\
R_t &= \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta \\
C_t^{-\eta} &= \beta E_t \left(C_{t+1}^{-\eta} R_{t+1} \right) \\
\frac{Y_t}{N_t} &= \frac{b}{1-\alpha} C_t^{\eta}
\end{aligned}$$

  and the process for the technology variable

$$\log A_t = (1-\rho) \log A^* + \rho \log A_{t-1} + \epsilon_t$$

- These are not a set of linear difference equations, but a mix of both linear and nonlinear equations: We need to figure out how to get a solution or at least approximate one.

Part II

Log-Linearization

Linearization

- In general, nonlinear systems like this cannot be solved analytically. However, it turns out their solution can be very well approximated by a corresponding set of linear equations.
- The idea is to use Taylor series approximations. In general, any nonlinear function $F(x_t, y_t)$ can be approximated around any point (x_t^*, y_t^*) using the formula

$$\begin{aligned} F(x_t, y_t) &= F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) \\ &\quad + F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t - x_t^*)(y_t - y_t^*) \\ &\quad + F_{yy}(x_t^*, y_t^*)(y_t - y_t^*)^2 + \ldots \end{aligned}$$

- If the gap between (x_t, y_t) and (x_t^*, y_t^*) is small, then terms in second and higher powers and cross-terms will all be very small and can be ignored, leaving something like

$$F(x_t, y_t) \approx \alpha + \beta_1 x_t + \beta_2 y_t$$

- But if we "linearize" around a point that (x_t, y_t) is far away from, then this approximation will not be accurate.

Log-Linearization

- DSGE models use a particular version of this technique. They take logs and then linearize the logs of variables around a simple "steady-state" path in which all real variables are growing at the same rate.
- The steady-state path is relevant because the stochastic economy will, on average, tend to fluctuate around the values given by this path, making the approximation an accurate one.
- This gives us a set of linear equations in the deviations of the logs of these variables from their steady-state values.
- Remember that log-differences are approximately percentage deviations

$$\log X - \log Y \approx \frac{X - Y}{Y}$$

  so this approach gives us a system that expresses variables in terms of their percentage deviations from the steady-state paths. It can be thought of as giving a system of variables that represents the business-cycle component of the model. Coefficients are elasticities and IRFs are easy to interpret.
- Also log-linearization is easy. It doesn't require taking lots of derivatives.

How Log-Linearization Works

- We will use lower-case letters to define log-deviations of variables from their steady-state values.

$$x_t = \log X_t - \log X^*$$

- The key to the log-linearization method is that every variable can be written as

$$X_t = X^* \frac{X_t}{X^*} = X^* e^{x_t}$$

- The big trick is that a first-order Taylor approximation of e^{x_t} is given by

$$e^{x_t} \approx 1 + x_t$$

- So, we can write variables as

$$X_t \approx X^* \left(1 + x_t\right)$$

- The second trick is for variables multiplying each other such as

$$X_t Y_t \approx X^* Y^* \left(1 + x_t\right)\left(1 + y_t\right) \approx X^* Y^* \left(1 + x_t + y_t\right)$$

  i.e. you set terms like $x_t y_t = 0$ because we are looking at small deviations from steady-state and multiplying these small deviations together one gets a term close to zero.

Anything Else?

- No, that's it.
- Substitute these approximations for the variables in the model, lots of terms end up canceling out, and when you're done you've got a system in the deviations of logged variables from their steady-state values.
- The best way to understand this stuff is to see it at work, so let's work through some examples from the RBC model.
- Note that we have assumed that technology (the source of all long-run growth in this economy) is given by

$$a_t = \rho a_{t-1} + \epsilon_t$$

  so there is no trend growth in this economy.
- This means that the steady-state variables are all constants. Technically, there is no great difficulty in modelling an economy with trend growth but this case is a bit simpler.

Log-Linearization Example 1

- Start with

$$Y_t = C_t + I_t$$

- Re-write it as

$$Y^* e^{y_t} = C^* e^{c_t} + I^* e^{i_t}$$

- Using the first-order approximation, this becomes

$$Y^* (1 + y_t) = C^* (1 + c_t) + I^* (1 + i_t)$$

- Note, though, that the steady-state terms must obey identities so

$$Y^* = C^* + I^*$$

- Canceling these terms on both sides, we get

$$Y^* y_t = C^* c_t + I^* i_t$$

which we will write as

$$y_t = \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t$$

Log-Linearization Example 2

- Now consider
$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$
- This can be re-written in terms of steady-states and log-deviations as
$$Y^* e^{y_t} = \left(A^* e^{a_t}\right)\left(K^*\right)^{\alpha} e^{\alpha k_{t-1}} \left(N^*\right)^{1-\alpha} e^{(1-\alpha) n_t}$$
- Again, use the fact the steady-state values obey identities so that
$$Y^* = A^* \left(K^*\right)^{\alpha} \left(N^*\right)^{1-\alpha}$$
- So canceling gives
$$e^{y_t} = e^{a_t} e^{\alpha k_{t-1}} e^{(1-\alpha) n_t}$$
- Using first-order Taylor approximations, this becomes
$$(1 + y_t) = (1 + a_t)(1 + \alpha k_{t-1})(1 + (1-\alpha) n_t)$$
- Ignoring cross-products of the log-deviations, this simplifies to
$$y_t = a_t + \alpha k_{t-1} + (1-\alpha) n_t$$

The Full Log-Linearized System

Once all the equations have been log-linearized, we have a system of seven equations of the form

$$
\begin{aligned}
y_t &= \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t \\
y_t &= a_t + \alpha k_{t-1} + (1-\alpha) n_t \\
k_t &= \frac{I^*}{K^*} i_t + (1-\delta) k_{t-1} \\
n_t &= y_t - \eta c_t \\
c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
r_t &= \left(\frac{\alpha}{R^*} \frac{Y^*}{K^*} \right) (y_t - k_{t-1}) \\
a_t &= \rho a_{t-1} + \epsilon_t
\end{aligned}
$$

We are nearly ready to put the model on the computer. However, notice that three of the equations have coefficients that are values relating to the steady-state path. These need to be calculated.

Part III

Calculating the Steady-State

The Steady-State Interest Rate

- We need to calculate $\frac{C^*}{Y^*}$, $\frac{I^*}{K^*}$ and $\frac{\alpha}{R^*}\frac{Y^*}{K^*}$
- We do this by taking the original non-linearized RBC system and figuring out what things look like along a zero growth path.
- Start with the steady-state interest rate. This is linked to consumption behaviour via the so-called Euler equation (or Keynes-Ramsey condition):

$$1 = \beta E_t \left(\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right)$$

- Because we have no trend growth in technology in our model, the steady-state features consumption, investment, and output all taking on constant values with no uncertainty.
- Thus, in steady-state, we have $C_t^* = C_{t+1}^* = C^*$, so

$$R^* = \beta^{-1}$$

In a no-growth economy, the rate of return on capital is determined by the rate of time preference.

Other Steady-State Calculations

- Take the equation for the rate of return on capital

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta$$

- In steady-state, we have

$$R^* = \beta^{-1} = \alpha \frac{Y^*}{K^*} + 1 - \delta$$

- So, in steady-state, we have

$$\frac{Y^*}{K^*} = \frac{\beta^{-1} + \delta - 1}{\alpha}$$

- Together with the steady-state interest equation, this tells us that

$$\frac{\alpha}{R^*} \frac{Y^*}{K^*} = \alpha\beta \left(\frac{\beta^{-1} + \delta - 1}{\alpha} \right) = 1 - \beta (1 - \delta)$$

which is the one of the steady-state values required

Investment-Capital and Investment-Output Ratios

- Next, we use the identity

$$K_t = I_t + (1 - \delta) K_{t-1}$$

- And the fact that in steady-state we have $K_t^* = K_{t-1}^* = K^*$, to give

$$\frac{I^*}{K^*} = \delta$$

  which was also required.

- This can then be combined with the previous steady-state ratio to give

$$\frac{I^*}{Y^*} = \frac{\frac{I^*}{K^*}}{\frac{Y^*}{K^*}} = \frac{\alpha\delta}{\beta^{-1} + \delta - 1}$$

- And obviously

$$\frac{C^*}{Y^*} = 1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1}$$

  which gives us the other required steady-state ratios.

The Final System

Using these steady-state identities, our system becomes

$$
\begin{aligned}
y_t &= \left(1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}\right) c_t + \left(\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right) i_t \\
y_t &= a_t + \alpha k_{t-1} + (1-\alpha)\, n_t \\
k_t &= \delta i_t + (1-\delta)\, k_{t-1} \\
n_t &= y_t - \eta c_t \\
c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
r_t &= (1-\beta(1-\delta))\,(y_t - k_{t-1}) \\
a_t &= \rho a_{t-1} + \epsilon_t
\end{aligned}
$$

This is written in the standard format for systems of linear stochastic difference equations. So, once we make assumptions about the underlying parameter values $(\alpha, \beta, \delta, \eta, \rho)$ we can apply solution algorithms such as Chris Sims's `gensys.m` routine to obtain a reduced-form solution, and thus simulate the model on the computer.

Part IV

Simulating the Model

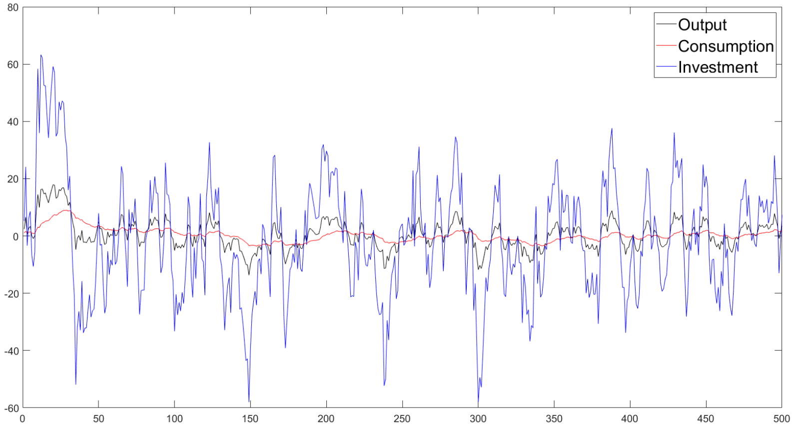
Parameterizing, Simulating and Checking IRFs

- The next page shows how to specify, solve and simulate the RBC model using Dynare, a package that works with Matlab. Note that you can write the model in intuitive form and Dynare will do the log-linearising and solving for steady states as well as simulating the model and calculating IRFs.
- The subsequent few pages show some charts that illustrate the properties of this model.
- The uses parameter values intended for analysis of quarterly time series: $\alpha = \frac{1}{3}$, $\beta = 0.99$, $\delta = 0.015$, $\rho = 0.95$, and $\eta = 1$ (i.e. log preferences).
- The first chart shows results from a 500-period simulation of this model. It demonstrates the main successful feature of the RBC model: It generates actual business cycles and they don't look too unrealistic.
- In particular, reasonable parameterizations of the model can roughly match the magnitude of observed fluctuations in output, and the model can match the fact that investment is far more volatile than consumption.
- In the early days of RBC research, this ability to match business cycle dynamics was considered a major strength, and many economists began to claim that there was no need for market imperfections to explain business cycles.

Dynare Code Specifying, Solving and Simulating the RBC Model

```
// This is a Dynare mod file for simple RBC model
// endogenous variables:
var y, c, k, n, r, a, i;
// shocks
varexo e;

parameters alpha, rho, beta, delta, eta;

alpha = 0.33;
rho   = 0.90;
beta  = 0.99;
delta = .015;
eta   = 1;
abar  = 1;

model;
    y = a*(k(-1)^alpha)*(n^(1-alpha)) ;
    k = (1-delta)*k(-1) + y - c ;
    r = alpha*(y/k(-1)) + 1 - delta;
    c^(-eta) =  beta*r(+1)*c(+1)^(-eta);
    y = ( c^eta / (1-alpha) )*n ;
    log(a) = rho*log(a(-1)) + e;
    i = y - c;
end;
initval;
c = 2;
k = 5;
n = 1;
r = 0.1;
a = 1;
e = 0;
end;

shocks;
var e;
stderr 1.;
end;

steady;

stoch_simul(periods=500, drop=200, order=1,irf=50, loglinear)  y, c, k, n, r, a, i;
```

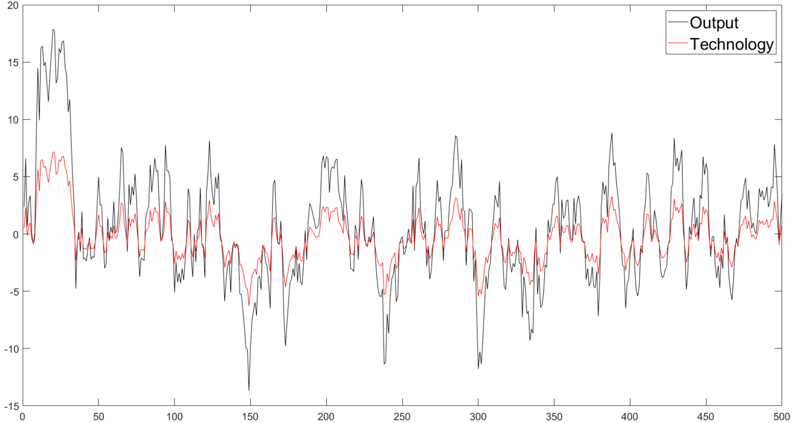
RBC Models Can Generate Cycles with Volatile Investment



The RBC Model's Propagation Mechanisms

- Despite this success, these RBC models have still come in for some criticism.
- One reason is that they have not quite lived up to the hype of their early advocates. Part of that hype stemmed from the idea that RBC models contained important *propagation mechanisms* for turning technology shocks into business cycles.
- The idea was that increases in technology induced extra output through higher capital accumulation and by inducing people to work more.
- In other words, some of the early research suggested that even in a world of iid technology levels, one would expect RBC models to still generate business cycles.
- However, the figure on the next page shows that output fluctuations in this model follow technology fluctuations quite closely: This shows that these additional propagation mechanisms are quite weak.

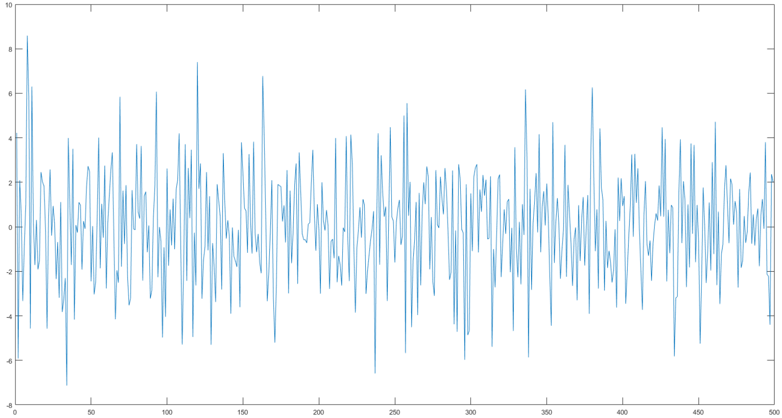
RBC Cycles Rely Heavily on Technology Fluctuations



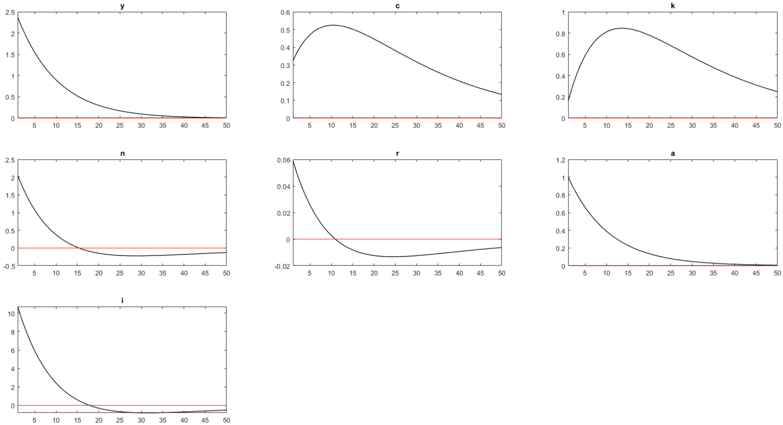
Autocorrelated Growth and Hump-Shaped IRFs

- Cogley and Nason (*AER*, 1995) noted another fact about business cycles that the RBC model does not match: Output growth is positively autocorrelated (not very—autocorrelation coefficient of 0.34—but statistically significant).
- But RBC models do not generate this pattern: See the figure on the next page. They can only do so if one simulates a technology process that has a positively autocorrelated growth rate.
- Cogley and Nason relate this back to the IRFs generated by RBC models. The figure on page 36 shows the responses of output, consumption, investment, and hours to a unit shock to ϵ_t.
- The figure on page 37 highlights that the response of output to the technology shock pretty much matches the response of technology itself.
- Cogley-Nason argue that one needs instead to have "humped-shaped" responses to shocks—a growth rate increase needs to be followed by another growth rate increase—if a model is to match the facts about autocorrelated output growth. The responses to technology shocks do not deliver this. Also, while we don't have other shocks in the model (e.g. government spending shocks), Cogley-Nason show RBC models don't generate hump-shaped responses for these either.

RBCs Do Not Generate Positively Autocorrelated Growth



Impulse Response Functions to Technology Shock



Extending the RBC Approach

- In addition to the Cogley-Nason critique, RBC models have also been criticised for failing to explain the labour market response to technology shocks.
- For example, a well known paper by Jordi Gali used long-run restriction identification VAR methods to show that hours worked tends to decline after a positive technology shock in strong contrast to the model's predictions.
- Over the years, many different branches of research have worked on fixing the deficiencies of the basic RBC approach.
- Some of them involve putting extra bells and whistles on the basic market-clearing RBC approach: Examples include variable utilization, lags in investment projects, habit persistence in consumer utility. Adding these elements tends to strengthen the propagation mechanism element of the model.
- The second approach is to depart more systematically from the basic RBC approach by adding rigidities such as sticky prices and wages. Some papers do this and add the other bells-and-whistles. We will introduce a "full blown" DSGE model of this type next.

Part V

The Smets-Wouters Model

A Popular DSGE Model

- Now we will discuss a paper presenting a modern DSGE model that has a number of New-Keynesian features and which has been estimated with Bayesian methods.
- The paper is "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach" by Frank Smets and Raf Wouters which was published in the *American Economic Review* in 2007.
- Smets is an economist with the ECB and Wouters works for the National Bank of Belgium and the model was first developed for the euro area. Models like this have been used for policy analysis at the ECB and other central banks.
- This paper estimated the model for US data.
- Both the euro area and U.S. Smets-Wouters papers have been among the most cited papers in economics in recent years.
- We will first present the log-linearized version of the model. An appendix with the full model is available on the class website.
- We will then discuss various applications of the model.

The Log-Linearized Model: The Supply Side

- The aggregate production function is

$$y_t = \phi_p \left(\alpha k_t^s + (1-\alpha) l_t + \epsilon_t^a \right)$$

where y_t is GDP, l_t is labour input, ϵ_t^a is total factor productivity and k_t^s is capital in use, which is determined by the amount of capital installed in the previous period and a capacity utilisation variable

$$k_t^s = k_{t-1} + z_t$$

- There are cost of adjusting the amount of capital in use so optimisation conditions for producers mean the rate of capacity utilisation is linked to the marginal productivity of capital

$$z_t = z_1 r_t^k$$

- The marginal productivity of capital is a function of the capital-labour ratio and the real wage

$$r_t^k = -\left(k_t - l_t \right) + w_t$$

- Total factor productivity evolves over time according to

$$\epsilon_t^a = \rho_a \epsilon_{t-1}^a + \eta_t^a$$

The Log-Linearized Model: The Demand Side

- The expenditure formulation of the aggregate resource constraint is

$$y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g$$

  where y_t is GDP, c_t is consumption, i_t is investment and ϵ_t^g is exogenous spending. (Terms like c_y and i_y are constant parameters here.)
- The variable z_t features here because we are assuming there are costs associated with having high rates of capacity utilisation.
- Exogenous spending is assumed to have two components: Government spending and element related to productivity because "net exports may be affected by domestic productivity developments."
- Taken together, exogenous spending changes over time according to

$$\epsilon_t^g = \rho \epsilon_{t-1}^g + \eta_t^g + \rho_{ga} \eta_t^a$$

The Log-Linearized Model: Consumption

- Consumption is determined by

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (l_t - E_t l_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \epsilon_t^b \right)$$

  where c_1, c_2, c_3 are constant parameters, r_t is the interest rate on a one-period safe bond and ϵ_t^b evolves according to

$$\epsilon_t^b = \rho_b \epsilon_{t-1}^b + \eta_t^b$$

- There are a number of aspects to this equation
  1. It is a consumption Euler equation with a backward-looking element added to it. This represents "habit formation" so that a term of the form $C_t - \lambda C_{t-1}$ replaces C_t in the utility function.
  2. The term involving labour input allows for some substitution between consumption and labour input.
  3. The coefficients c_1, c_2, c_3 are themselves functions of deeper structural parameters.
  4. Smets-Wouters describe the ϵ_b term as a "risk premium" shock determining the willingness of households to hold the one-period bond. It can also be seen as a type of preference shock that influences the short-term consumption-saving decision.

The Log-Linearized Model: Investment

- Investment is determined by

$$i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t + \epsilon_t^i$$

  where

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) r_{t+1}^k - \left(r_t - E_t \pi_{t+1} + \epsilon_t^b\right)$$

  and

$$k_t = k_1 k_{t-1} + (1 - k_1) i_t + k_2 \epsilon_t^i$$

- Again, there is quite a lot going on here
  1. Investment depends on lagged on investment because there is an adjustment cost function that limits that amount of new investment that can come "on line" immediately.
  2. The main driving force behind investment is q_t which itself is determined by a forward-looking stochastic difference equation.
  3. Solving the q_t equation would show that q_t depends positively on expected future marginal productivities of capital and negatively on future real interest rate (and "risk premia")
  4. The positive shock to investment also boosts the capital stock (representing "more productive" capital).

The Log-Linearized Model: Prices

- The mark-up of price over marginal cost is determined by

$$\mu_t^p = \alpha \left(k_t - l_t \right) + \epsilon_t^a - w_t$$

which factors in diminishing marginal productivity of capital, the effects of the productivity shock on costs and the real wage.

- Price inflation is then determined by

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

where ϵ_t^p is a price mark-up disturbance that evolves according to

$$\epsilon_t^p = \rho^p \epsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p$$

- Observations:
  - This is a New-Keynesian Phillips curve amended to provide a role for lagged inflation. This is modelled in the paper via the assumption that most firms index their price to past inflation and only occasionally get to set an optimal price.
  - The mark-up shock affects both current and lagged inflation in an attempt to get at temporary price level shocks.

The Log-Linearized Model: Wages

- The model treats wages similarly to prices, with sticky wages that gradually adjust so that real wages are move to equate the marginal costs and benefits of working.
- Specifically, wages move over time to equate real wages with the marginal rate of substitution between working and consuming. The gap between these is the "wage mark-up" defined as

$$\begin{aligned}\mu_t^w &= w_t - mrs_t \\ &= w_t - \left(\sigma l_t - \frac{1}{1-\lambda/\gamma}\left(c_t - \lambda c_{t-1}\right)\right)\end{aligned}$$

- Wages are then given by

$$w_t = w_1 w_{t-1} + \left(1 - w_1\right) E_t\left(w_{t+1} + \pi_{t+1}\right) - w_2 \pi_t + w_3 \pi_{t-1} - w_t \mu_t^w + \epsilon_t^w$$

where

$$\epsilon_t^w = \rho^w \epsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$$

The Log-Linearized Model: Monetary Policy

- The final element of the model is a rule for monetary policy. It is assumed that the central bank sets short-term interest rates according to

$$\begin{aligned} r_t &= \rho r_{t-1} + (1-\rho)\left(r_\pi \pi_t + r_y \left(y_t - y_t^p\right)\right) \\ &\quad + r_{\Delta y}\left[\left(y_t - y_t^p\right) - \left(y_{t-1} - y_{t-1}^p\right)\right] + \epsilon_t^r \end{aligned}$$

  where

$$\epsilon_t^r = \rho^r \epsilon_{t-1}^r + \eta_t^r$$

- Here the interest rate depends on last period's interest rate while gradually adjusting towards a target interest rate $\left(r_\pi \pi_t + r_y \left(y_t - y_t^p\right)\right)$ that depends on inflation and the gap between output and its potential level $\left(y_t - y_t^p\right)$. It also depends on the growth rate of this output gap.
- Potential output is defined as the level of output that would prevail if prices and wages were fully flexible. This means the model effectively needs to be "expanded" to add a "shadow" flexible-price economy.

Why So Many Bells And Whistles?

- Relative to the pure RBC or New Keynesian models we saw before, this model has lots of additional features:
  1. Adjustment costs for investment.
  2. Capacity utilisation costs.
  3. Habit persistence.
  4. Price indexation.
  5. Wage indexation.
  6. Lots of new autocorrelated distubance terms.
- These help the model to address the weaknesses of the previous models.
  1. Adjustment costs, utilisation costs and habit persistence all help to "throw sand in wheels" of the model, making variables more sluggish and giving random shocks a more long-lasting effect. This was a weakness of the RBC model.
  2. Indexation deals with the NK model's failure to match inflation persistence.
- Still, it is hard to argue these are really "micro-founded" mechanisms. In many ways, the model is quite ad hoc and hardly immune to the Lucas critique.

The Observable VAR System

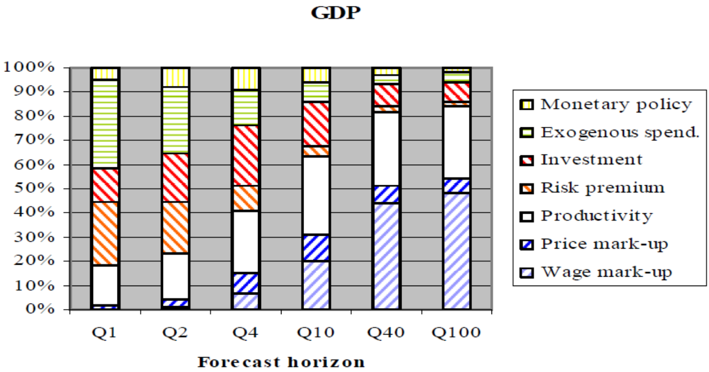
$$
(15) \qquad Y_t = \begin{bmatrix} dlGDP_t \\ dlCONS_t \\ dlINV_t \\ dlWAG_t \\ lHOURS_t \\ dlP_t \\ FEDFUNDS_t \end{bmatrix} = \begin{bmatrix} \bar{\gamma} \\ \bar{\gamma} \\ \bar{\gamma} \\ \bar{\gamma} \\ \bar{l} \\ \bar{\pi} \\ \bar{r} \end{bmatrix} + \begin{bmatrix} y_t - y_{t-1} \\ c_t - c_{t-1} \\ i_t - i_{t-1} \\ w_t - w_{t-1} \\ l_t \\ \pi_t \\ r_t \end{bmatrix},
$$

Out-of-Sample Forecasting Beats VAR Models

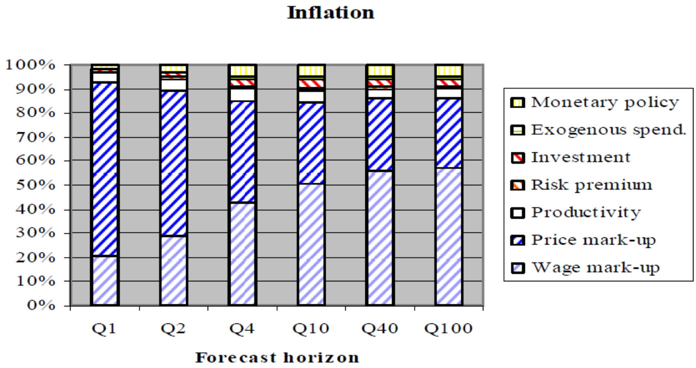
TABLE 3—OUT-OF-SAMPLE PREDICTION PERFORMANCE

	GDP	dP	Fedfunds	Hours	Wage	CONS	INV	Overall
VAR(1)	*RMSE-statistic for different forecast horizons*							
1q	0.60	0.25	0.10	0.46	0.64	0.60	1.62	−12.87
2q	0.94	0.27	0.18	0.78	1.02	0.95	2.96	−8.19
4q	1.64	0.34	0.36	1.45	1.67	1.54	5.67	−3.25
8q	2.40	0.53	0.64	2.13	2.88	2.27	8.91	1.47
12q	2.78	0.63	0.79	2.41	4.09	2.74	10.97	2.36
BVAR(4)	*Percentage gains* (+) *or losses* (−) *relative to VAR*(1) *model*							
1q	2.05	14.14	−1.37	−3.43	2.69	12.12	2.54	3.25
2q	−2.12	15.15	−16.38	−7.32	−0.29	10.07	2.42	0.17
4q	−7.21	31.42	−12.61	−8.58	−3.82	1.42	0.43	0.51
8q	−15.82	33.36	−13.26	−13.94	−8.98	−8.19	−11.58	−4.10
12q	−15.55	37.59	−13.56	−4.66	−15.87	−3.10	−23.49	−9.84
DSG	*Percentage gains* (+) *or losses* (−) *relative to VAR*(1) *model*							
1q	5.68	2.05	−8.24	0.68	5.99	20.16	9.22	3.06
2q	14.93	10.62	−17.22	10.34	6.20	25.85	16.79	2.82
4q	20.17	46.21	1.59	19.52	9.21	26.18	21.42	6.82
8q	22.55	68.15	28.33	22.34	15.72	21.82	25.95	11.50
12q	32.17	74.15	40.32	27.05	21.88	23.28	41.61	13.51

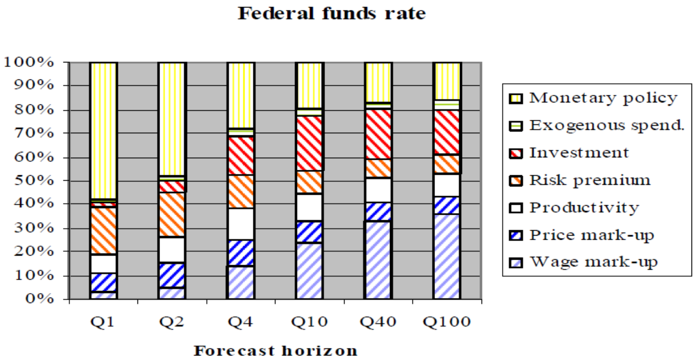
Explaining GDP Movements At Various Horizons



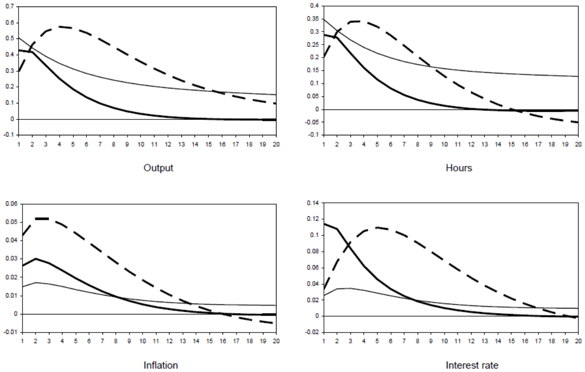
Explaining Inflation Movements At Various Horizons



Explaining Fed Funds Movements At Various Horizons

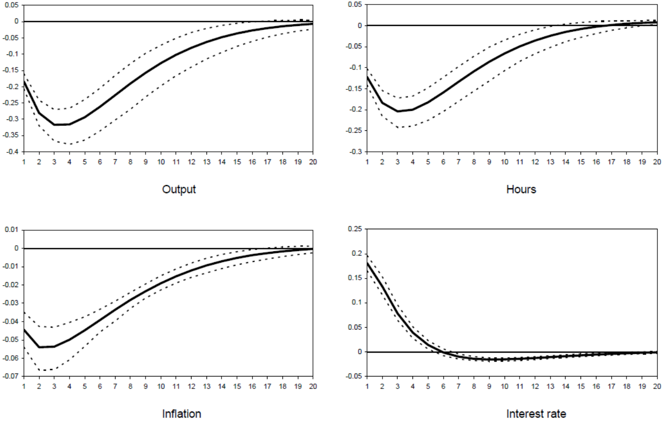


The Impact of Various “Demand” Shocks

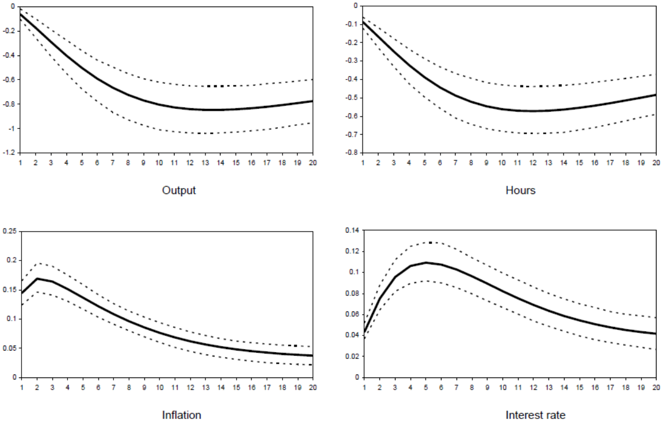


Notes: Bold solid line: risk premium shock; thin solid line: exogenous spending shock; dashed line: investment shock.

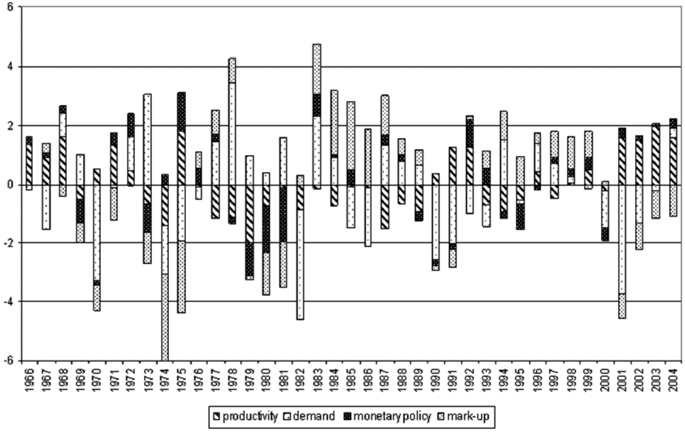
Impulse Response for a Monetary Policy Shock



Impulse Response for a Wage Mark-Up Shock



Decomposing the Growth Rate of GDP



Weaknesses and Strengths of DSGE Models

- You've seen enough now to have a good sense of what modern DSGE models look like and what they are used for.
- The following is a fair list of weaknesses of these models
  1. A large number of *ad hoc* economic mechanisms designed mainly to fit persistence properties of the data rather than because economists have a strong belief in these particular stories.
  2. A large amount of unexplained shocks which are often highly persistent.
  3. A minimal treatment of banking and financial markets (still true despite current ongoing work.)
  4. Very limited modelling of policy tools or details of national accounts.
  5. Plenty of evidence that pure rational expectations assumption is flawed.
  6. Claims that they are based on stable structural parameters and thus immune to the Lucas critique are silly.
- Still, there are a number of positive aspects that don't feature in VARs (imposition of budget constraints, a consistent story for how agents behave and a coherent handling of expectations) and these strengths may help DSGEs to be more useful for forecasting and "what if" analysis than VARs.