

MA Advanced Macroeconomics

Vector Autoregressions: Methods and Examples

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Background on VARs

- Introduced by Christopher Sims (1980) in a path-breaking article titled “Macroeconomics and Reality.”
- Sims was indeed telling the macro profession to “get real.”
- He criticized the widespread use of highly specified macro-models that made very strong identifying restrictions (in the sense that each equation in the model usually excluded most of the model’s other variables from the right-hand-side) as well as very strong assumptions about the dynamic nature of these relationships.
- VARs were an alternative that allowed one to model macroeconomic data accurately, without having to impose lots of incredible restrictions: “macro modelling without pretending to have too much a priori theory.”
- We will see that VARs are not theory free. But they do make the role of theoretical identifying assumptions far clearer than was the case for the types of models Sims was criticizing.

Matrix Formulation of VARs

- The simplest possible VAR features two variables and one lag:

$$y_{1t} = a_{11}y_{1,t-1} + a_{12}y_{2,t-1} + e_{1t}$$

$$y_{2t} = a_{21}y_{1,t-1} + a_{22}y_{2,t-1} + e_{2t}$$

- The most compact way to express a VAR system like this is to use matrices. Defining the matrices

$$Y_t = \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$e_t = \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}$$

- This system can be written as

$$Y_t = AY_{t-1} + e_t$$

Vector Moving Average (VMA) Representation

- VARs express variables as function of what happened yesterday and today's shocks.
- But what happened yesterday depended on yesterday's shocks and on what happened the day before.
- This VMA representation is obtained as follows

$$\begin{aligned} Y_t &= e_t + AY_{t-1} \\ &= e_t + A(e_{t-1} + AY_{t-2}) \\ &= e_t + Ae_{t-1} + A^2(e_{t-2} + AY_{t-3}) \\ &= e_t + Ae_{t-1} + A^2e_{t-2} + A^3e_{t-3} + \dots + A^te_0 \end{aligned}$$

- This makes clear how today's values for the series are the cumulation of the effects of all the shocks from the past.
- It is also useful for deriving predictions about the properties of VARs.

Impulse Response Functions

- Suppose there is an initial shock defined as

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and then all error terms are zero afterwards, i.e. $e_t = 0$ for $t > 0$.

- Recall VMA representation

$$Y_t = e_t + Ae_{t-1} + A^2e_{t-2} + A^3e_{t-3} + \dots + A^te_0$$

- This tells us that the response after n periods is $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- So IRFs for VARs are directly analogous to the IRFs for AR(1) models that we looked at last week.

Using a VAR to Forecast

- VARs are often used for forecasting.
- Suppose we observe our vector of variables Y_t . What's our forecast for Y_{t+1} ?
- The model for next period is

$$Y_{t+1} = AY_t + e_{t+1}$$

- Because $E_t e_{t+1} = 0$, an unbiased forecast at time t is AY_t . In other words, $E_t Y_{t+1} = AY_t$.
- The same reasoning tells us that $A^2 Y_t$ is an unbiased forecast of Y_{t+2} and $A^3 Y_t$ is an unbiased forecast of Y_{t+3} and so on.
- So once we've estimated a VAR of this form, they are very easy to construct forecasts from.

Generality of the First-Order Matrix Formulation: I

- The model we've been looking at may seem like a small subset of all possible VARs because it doesn't have a constant term and only has lagged values from one period ago.
- However, one can add a third variable here which takes the constant value 1 each period. The equation for the constant term will just state that it equals its own lagged values. So this formulation actually incorporates models with constant terms.
- We would also expect most equations in a VAR to have more than one lag. Surely this makes things much more complicated?
- Not really. It turns out, the first-order matrix formulation can represent VARs with longer lags.
- Consider the two-lag system

$$y_{1t} = a_{11}y_{1,t-1} + a_{12}y_{1,t-2} + a_{13}y_{2,t-1} + a_{14}y_{2,t-2} + e_{1t}$$

$$y_{2t} = a_{21}y_{1,t-1} + a_{22}y_{1,t-2} + a_{23}y_{2,t-1} + a_{24}y_{2,t-2} + e_{2t}$$

Generality of the First-Order Matrix Formulation: II

- Now define the vector

$$Z_t = \begin{pmatrix} y_{1t} \\ y_{1,t-1} \\ y_{2t} \\ y_{2,t-1} \end{pmatrix}$$

- This system can be represented in matrix form as

$$Z_t = AZ_{t-1} + e_t$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad e_t = \begin{pmatrix} e_{1t} \\ 0 \\ e_{2t} \\ 0 \end{pmatrix}$$

- This is sometimes called the “companion form” matrix formulation.

Interpreting Shocks and Impulse Responses

- The system we've been looking at is usually called a *reduced-form* VAR model.
- It is a purely *econometric* model, without any theoretical element.
- How should we interpret it? One interpretation is that e_{1t} is a shock that affects only y_{1t} on impact and e_{2t} is a shock that affects only y_{2t} on impact.
- For instance, one can use the IRFs generated from an inflation-output VAR to calculate the dynamic effects of “a shock to inflation” and “a shock to output”.
- But other interpretations are available.
- For instance, one might imagine that the true shocks generating inflation and output are an “aggregate supply” shock and an “aggregate demand” shock and that both of these shocks have a direct effect on both inflation and output.
- How would we identify these “structural” shocks and their impulse responses?

The Multiplicity of Shocks and IRFs

- Suppose reduced-form and structural shocks are related by

$$e_{1t} = c_{11}\epsilon_{1t} + c_{12}\epsilon_{2t}$$

$$e_{2t} = c_{21}\epsilon_{1t} + c_{22}\epsilon_{2t}$$

- Can write this in matrix form as

$$e_t = C\epsilon_t$$

- These two VMA representations describe the data equally well:

$$\begin{aligned} Y_t &= e_t + Ae_{t-1} + A^2e_{t-2} + A^3e_{t-3} + \dots + A^te_0 \\ &= C\epsilon_t + AC\epsilon_{t-1} + A^2C\epsilon_{t-2} + A^3C\epsilon_{t-3} + \dots + A^tC\epsilon_0 \end{aligned}$$

- Can interpret the model as one with shocks e_t and IRFs given by A^n .
- Or as a model with structural shocks ϵ_t and IRFs are given by $A^n C$.
- And we could do this for any C : We just don't know the structural shocks.

Contemporaneous Interactions: I

- Another way to see how reduced-form shocks can be different from structural shocks is if there are contemporaneous interactions between variables, which is likely.
- Consider the following model:

$$y_{1t} = a_{12}y_{2t} + b_{11}y_{1,t-1} + b_{12}y_{2,t-1} + \epsilon_{1t}$$

$$y_{2t} = a_{21}y_{1t} + b_{21}y_{1,t-1} + b_{22}y_{2,t-1} + \epsilon_{2t}$$

- Can be written in matrix form as

$$AY_t = BY_{t-1} + \epsilon_t$$

where

$$A = \begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix}$$

Contemporaneous Interactions: II

- Now if we estimate the “reduced-form” VAR model

$$Y_t = DY_{t-1} + e_t$$

- Then the reduced-form shocks and coefficients are

$$D = A^{-1}B$$

$$e_t = A^{-1}\epsilon_t$$

- Again, the following two decompositions both describe the data equally well

$$\begin{aligned} Y_t &= e_t + De_{t-1} + D^2e_{t-2} + D^3e_{t-3} + \dots \\ &= A^{-1}\epsilon_t + DA^{-1}\epsilon_{t-1} + D^2A^{-1}\epsilon_{t-2} + \dots + D^tA^{-1}\epsilon_0 \end{aligned}$$

- For the structural model, the impulse responses to the structural shocks from n periods are given by D^nA^{-1} .

Structural VARs: A General Formulation

- In its general formulation, the structural VAR is

$$AY_t = BY_{t-1} + C\epsilon_t$$

- The model is fully described by the following parameters:
 - ① n^2 parameters in A
 - ② n^2 parameters in B
 - ③ n^2 parameters in C
 - ④ $\frac{n(n+1)}{2}$ parameters in Σ , which describes the pattern of variances in covariances underlying the shock terms.
- Adding all these together, we see that the most general form of the structural VAR is a model with $3n^2 + \frac{n(n+1)}{2}$ parameters.

Identification of Structural VARs: The General Problem

- Estimating the reduced-form VAR

$$Y_t = DY_{t-1} + e_t$$

gives us information on $n^2 + \frac{n(n+1)}{2}$ parameters: The coefficients in D and the estimated covariance matrix of the reduced-form errors.

- To obtain information about structural shocks, we thus need to impose $2n^2$ *a priori* theoretical restrictions on our structural VAR.
- This will leave us with $n^2 + \frac{n(n+1)}{2}$ known reduced-form parameters and $n^2 + \frac{n(n+1)}{2}$ structural parameters that we want to know.
- This can be expressed as $n^2 + \frac{n(n+1)}{2}$ equations in $n^2 + \frac{n(n+1)}{2}$ unknowns, so we can get a unique solution.
- Example: Asserting that the reduced-form VAR is the structural model is the same as imposing the $2n^2$ *a priori* restrictions that $A = C = I$.

Recursive SVARs

- SVARs often identify their shocks as coming from distinct independent sources.
- For instance, a pure “aggregate supply” or “technology” shock is usually seen as being completely independent from an “aggregate demand” or “preference” shock.
- But the error series in reduced-form VARs are usually correlated with each other. One way to view these correlations is that the reduced-form errors are combinations of a set of statistically independent structural errors.
- The most popular SVAR method is the recursive identification method. This method (used in the original Sims paper) uses simple regression techniques to construct a set of uncorrelated structural shocks directly from the reduced-form shocks.
- This method sets $A = I$ and constructs a C matrix so that the structural shocks will be uncorrelated.

The Cholesky Decomposition

- Start with a reduced-form VAR with three variables and errors e_{1t}, e_{2t}, e_{3t} .
- Take one of the variables and assert that this is the first structural shock, $\epsilon_{1t} = e_{1t}$.
- Then run the following two OLS regressions involving the reduced-form shocks

$$e_{2t} = c_{21}e_{1t} + \epsilon_{2t} \quad (1)$$

$$e_{3t} = c_{31}e_{1t} + c_{32}e_{2t} + \epsilon_{3t} \quad (2)$$

- This gives us a matrix equation $Ge_t = \epsilon_t$.
- Inverting G gives us C so that $e_t = C\epsilon_t$. Identification done.
- Remember that error terms in OLS equations are uncorrelated with the right-hand-side variables in the regressions.
- Note now that, by construction, the ϵ_t shocks constructed in this way are uncorrelated with each other.

Interpreting the Cholesky Decomposition

- The method posits a sort of “causal chain” of shocks.
- The first shock affects all of the variables at time t . The second only affects two of them at time t , and the last shock only affects the last variable at time t .
- The reasoning usually relies on arguments such as “certain variables are sticky and don’t respond immediately to some shocks.” We will discuss examples next week.
- A serious drawback: The causal ordering is not unique. Any one of the VARs variables can be listed first, and any one can be listed last.
- This means there are $n! = (1)(2)(3)\dots(n)$ possible recursive orderings.
- Which one you like will depend on your own prior thinking about causation.

Another Way to Do Recursive VARs

- The idea of certain shocks having effects on only some variables at time t can be re-stated as some *variables* only having effects on some variables at time t .
- In our 3 equation example this method sets $C = I$ and directly estimates the A and B matrices using OLS:

$$y_{1t} = b_{11}y_{1,t-1} + b_{12}y_{2,t-1} + b_{13}y_{3,t-1} + \epsilon_{1t}$$

$$y_{2t} = b_{21}y_{1,t-1} + b_{22}y_{2,t-1} + b_{23}y_{3,t-1} - a_{21}y_{1t} + \epsilon_{2t}$$

$$y_{3t} = b_{31}y_{1,t-1} + b_{32}y_{2,t-1} + b_{33}y_{3,t-1} - a_{31}y_{1t} - a_{32}y_{2t} + \epsilon_{3t}$$

- See how the first shock affects all the variables while the last shock only affects the last variable.
- This method delivers shocks and impulse responses that are identical to the Cholesky decomposition.
- Shows that different combinations of A , B and C can deliver the same structural model.

Two Examples of VAR Studies

We will look at two examples of studies that use recursive VARs:

- 1 Lutz Killian (2008): Not All Oil Price Shocks are Alike: Disentangling Demand and Supply Shocks in the Crude Oil Market.
- 2 James Stock and Mark Watson (2001), *Journal of Economic Perspectives*. This paper examines the effect of monetary policy shocks.

Killian on Oil Shocks

- Oil shocks—large run-ups and subsequent declines in the price of oil—regularly receive a lot of attention.
- Many recent recessions were preceded by an increase in the price of oil. Why exactly this has occurred is not obvious: Oil usage is actually a relatively small input compared to GDP.
- Previous empirical work has generally asked the question “what are the effects of an oil price shock?”
- Killian asks “what is an oil price shock and are there different kinds of oil price shocks?”
- He uses VAR analysis to distinguish between shocks to oil prices due to global demand, shocks due to oil supply, and shocks due to speculation in the oil price market.
- Let’s see how he does it.

The Model

- Three variable monthly VAR in the growth rate of oil production, real global economic activity, and the real price of oil:

$$z_t = (\Delta prod_t, rea_t, rpo_t)'$$

- VAR structure is

$$A_0 z_t = \alpha + \sum_{i=1}^{24} A_i z_{t-i} + \epsilon_t$$

where ϵ_t are the structural shocks, and A_0 is lower-rectangular

$$A_0 = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

- Identifying assumptions:
 - 1 Oil production does not respond within the month to world demand and oil prices
 - 2 World demand is affected within the month by oil production, but not by oil prices.
 - 3 Oil prices respond immediately to oil production and world demand.

Interpreting the Structural Shocks

- If A_0 is lower-triangular, then so is A_0^{-1} .
- Reduced-form model is

$$z_t = A_0^{-1}\alpha + \sum_{i=1}^{24} A_0^{-1}A_i z_{t-i} + A_0^{-1}\epsilon_t$$

- Reduced-form shocks e_t related to structural shocks as $e_t = A_0^{-1}\epsilon_t$:

$$\begin{pmatrix} e_t^{\Delta prod} \\ e_t^{rea} \\ e_t^{rpo} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \epsilon_t^{\Delta prod} \\ \epsilon_t^{rea} \\ \epsilon_t^{rpo} \end{pmatrix}$$

- The oil production reduced-form shock is a structural shock; the reduced-form economic activity shock combines the structural oil shock and the structural activity shock; the reduced-form oil price shock combines all three structural shocks.

Checking the Identification Restrictions

Relative to the general model

$$AY_t = BY_{t-1} + C\epsilon_t$$

where are our $2n^2 = 18$ identifying restrictions?

- 1 We set $C = I$ instead assuming contemporaneous interactions between variables: 9 restrictions.
- 2 Lower-diagonal assumption on A_0 : 3 zero restrictions.
- 3 Unit coefficient normalization on diagonal of A_0 : 3 restrictions.
- 4 Orthogonal structural shocks: 3 off-diagonal elements of Σ are zero.

Decomposing the Variables

- In addition to the standard impulse response analysis, Killian shows how the real price of oil can be decomposed into components related to these three shocks. How did he do this?
- Recall the VMA representation:

$$Y_t = \epsilon_t + A\epsilon_{t-1} + A^2\epsilon_{t-2} + A^3\epsilon_{t-3} + \dots + A^t\epsilon_0$$

- One can do this calculation three times, each time with only one type of shock “turned on” and the other set to zero. Adding these up, one will get the realized values of Y_t .
- Alternatively, one can do a dynamic simulation of the model

$$Y_t = AY_{t-1} + \epsilon_t$$

in each case letting the ϵ_t represent one of the realized historical shocks with the others set to zero.

Some of Killian's Findings

- 1 Despite getting a lot of attention, shocks to oil supply have limited effects on oil prices and have been of negligible importance in driving oil prices over time.
- 2 Both global demand and speculative oil price shocks can have significant effects on oil prices, but speculative oil price shocks have limited effects on global economic activity.
- 3 Speculative oil-market shocks have accounted for most of the month-to-month movements in oil prices.
- 4 But the steady increase in oil prices from 2000 onwards was almost solely due to strong global demand.
- 5 Main Lesson: How the economy reacts to an “oil price shock” will depend on the origins of that shock.
- 6 Helps to explain why the world economy survived the lastest big increase in oil prices without going into recession.

A Monetary Policy VAR

- Stock and Watson's 2001 *JEP* paper is a very useful introduction to VAR methods.
- The paper contains an important application: What are the effects of monetary policy shocks?
- Can think of these VARs as useful in two ways:
 - 1 From a scientific perspective: Monetary policy co-moves with lots of other macro variables. Only by identifying the structural or exogenous shocks to policy can we discover its true effects.
 - 2 From a policy perspective, help to answer the question “if I choose to raise interest rates by an extra quarter point today, what is likely to happen over the next year to inflation and output relative to the case where I keep rates unchanged?” Essentially, this is a question about impulse responses.

Stock and Watson's VAR

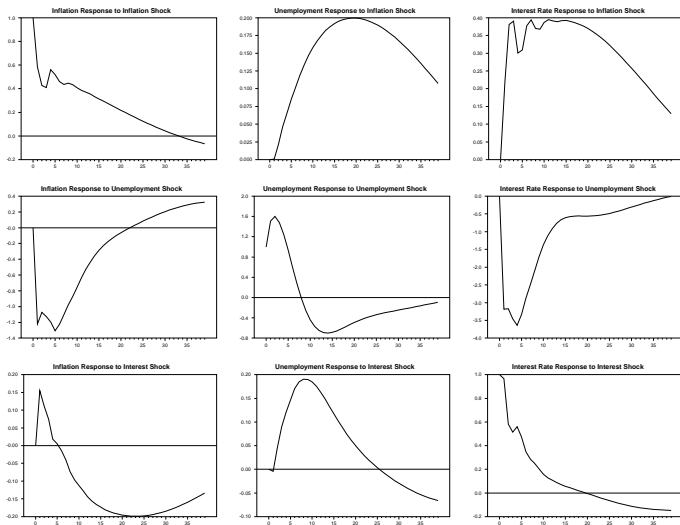
- Monthly data on inflation (π_t), the unemployment rate (u_t) and the federal funds rate (i_t).
- Posits a lower-triangular causal chain of the form

$$AZ_t = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \pi_t \\ u_t \\ i_t \end{pmatrix} = BZ_{t-1} + \epsilon_t$$

- Identifying assumptions
 - 1 Inflation depends only on lagged values of the other variables (sticky prices?)
 - 2 Unemployment depends on contemporaneous inflation but not the funds rate.
 - 3 The funds rate depends on both contemporaneous inflation and unemployment. (Fed using its knowledge about the current state of the economy when it is setting interest rates).
- Can you think of other identifying assumptions?

IRFs From Recursive VAR, First Identification

Order is Inflation, Unemployment, Interest Rate



IRFs From Recursive VAR, Second Identification

Order is Interest Rate, Unemployment, Inflation

