

Topic 1: The Solow Model of Economic Growth

About This Course

Although some of the topics we will cover will be familiar to you, the overall approach taken in this class will perhaps be more formal than you have seen before. We will tend to use a more mathematical approach to derive solutions to models and to characterise their properties. In some cases, this will involve introducing methods that you may not have seen before.

While this approach to macroeconomics may seem a little austere to some of you, it has some important advantages. For instance, a particular economic policy proposal might sound appealing, but an analytical examination could reveal drawbacks that are not clear from casual thinking. Writing down a formal economic model also allows one to be precise about the assumptions that need to be made to justify a particular policy proposal. Beyond the implications for applied policy analysis, the formal approach fits well with the modern econometric approach to testing economic theories. By providing explicit solutions for the determinants of various macroeconomic variables, this approach leads one more directly towards testable econometric equations. For those of you who intend to study more economics after this course, we hope to give you a flavour of the modern approach to macroeconomics, and perhaps teach you a few tools that may prove useful in the future.

Questions in Growth Theory and the Solow Model

We will spend the first part of this course on what is known as “growth theory.” This branch of macroeconomics concerns itself with big-picture questions: What determines the growth rate of the economy over the long run and what can policy measures do to affect it? This is, of course, related to the even more fundamental question of what makes some countries rich and others poor.

A useful starting point for illustrating the questions addressed by growth theory is the idea that output is produced using an aggregate production function technology. For illustration, assume that this takes the form of a constant returns to scale Cobb-Douglas production function:

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha} \quad 0 < \alpha < 1 \quad (1)$$

where K_t is capital input and L_t is labour input. Note that an increase in A_t results in higher output without having to raise inputs. Macroeconomists usually call increases in

A_t “technological progress” and sometimes I will loosely refer to this as the “technology” term, but ultimately A_t is simply a measure of productive efficiency. Because an increase in A_t increases the productiveness of the other factors, it is also sometimes known as *Total Factor Productivity* (TFP), and this is the term most commonly used in empirical papers that attempt to calculate this series.

Growth theory is primarily interested in the determination of output per person in the economy, rather than total output. For this reason, we will focus more on the determination of output per worker. This is obtained by dividing both sides of equation (1) by L_t to get

$$\frac{Y_t}{L_t} = A_t \left(\frac{K_t}{L_t} \right)^\alpha \quad (2)$$

This equation shows that, with a constant returns production function, there are two ways to increase output per worker:

- Capital deepening (i.e. increases in capital per worker)
- Technological progress: Improving the efficiency with which an economy uses its inputs.

One of the central question addressed by growth theory is the relative importance of these two sources of growth. This question is important because policies that focus on capital deepening (for instance, by tax policies aimed at boosting investment) are often likely to be quite different from policies that attempt to boost technological efficiency. Exactly what factors determine technological efficiency is another important question for growth theory and for the empirical study of economic growth.

An Alternative Expression for Output Per Worker

I also want to introduce an alternative characterisation of output per worker that turns out to be very useful. First, we’ll define the capital-output ratio as

$$x_t = \frac{K_t}{Y_t} \quad (3)$$

So, the production function can be expressed as

$$Y_t = A_t (x_t Y_t)^\alpha L_t^\beta \quad (4)$$

Here, we are using the fact that

$$K_t = x_t Y_t \quad (5)$$

Dividing both sides of this expression by Y_t^α , we get

$$Y_t^{1-\alpha} = A_t x_t^\alpha L_t^\beta \quad (6)$$

Taking both sides of the equation to the power of $\frac{1}{1-\alpha}$ we arrive at

$$Y_t = A_t^{\frac{1}{1-\alpha}} x_t^{\frac{\alpha}{1-\alpha}} L_t^{\frac{\beta}{1-\alpha}} \quad (7)$$

So, output per worker is

$$\frac{Y_t}{L_t} = A_t^{\frac{1}{1-\alpha}} x_t^{\frac{\alpha}{1-\alpha}} L_t^{\frac{\beta}{1-\alpha}-1} \quad (8)$$

If the economy has constant returns to scale, so that $\beta = 1 - \alpha$, this simplifies to

$$\frac{Y_t}{L_t} = A_t^{\frac{1}{1-\alpha}} x_t^{\frac{\alpha}{1-\alpha}} \quad (9)$$

This equation states that all fluctuations in output per worker are due to either changes in technological progress or changes in the capital-output ratio. When considering the relative role of technological progress or policies to encourage accumulation, we will see that this decomposition is more useful than equation (2) because the level of technology does not affect x_t in the long run while it does affect $\frac{K_t}{L_t}$. So, this decomposition offers a cleaner picture of the part of growth due to technology and the part that is not.

Some Mathematical Tricks

We are interested in modelling changes over time in outputs and inputs. A useful mathematical shorthand that saves us from having to write down derivatives with respect to time everywhere is to write

$$\dot{Y}_t = \frac{dY_t}{dt} \quad (10)$$

What we are really interested in, though, is *growth rates* of series: If I tell you GDP was up by 5 million euros, that may sound like a lot, but unless we scale it by the overall level of GDP, it's not really very useful information. Thus, what we are interested in calculating is $\frac{\dot{Y}_t}{Y_t}$, and this is our mathematical expression for the growth rate of a series.

Now, I'm going to introduce one of the techniques that we will use to obtain growth rates for variables of interest. This involves using logarithms. The reason for this is the following property:

$$\frac{d(\log Y_t)}{dt} = \frac{d(\log Y_t)}{dY_t} \frac{dY_t}{dt} = \frac{\dot{Y}_t}{Y_t} \quad (11)$$

The growth rate of a series is the same as the derivative of its log with respect to time (note the use of chain-rule of differentiation in the above equation.)

Two other useful properties of logarithms that will also help us characterise the dynamics of growth models are the following:

$$\log(XY) = \log X + \log Y \quad (12)$$

$$\log(X^Y) = Y \log X \quad (13)$$

To illustrate how to use the properties of logarithms to get growth rates, let's consider again the constant returns to scale Cobb-Douglas production function from equation (1). Taking logs of both sides of this equation, and then using the properties of the log function, we get

$$\log(Y_t) = \log(A_t K_t^\alpha L_t^{1-\alpha}) \quad (14)$$

$$= \log(A_t) + \log(K_t^\alpha) + \log(L_t^{1-\alpha}) \quad (15)$$

$$= \log(A_t) + \alpha \log(K_t) + (1 - \alpha) \log(L_t) \quad (16)$$

Now taking the derivative with respect to time, we get the required formula:

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} + \alpha \frac{\dot{K}_t}{K_t} + (1 - \alpha) \frac{\dot{L}_t}{L_t} \quad (17)$$

This takes us from the Cobb-Douglas formula involving levels to a simple formula involving growth rates. The growth rate of output per worker is simply

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{A}_t}{A_t} + \alpha \left(\frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} \right) \quad (18)$$

This is a re-statement in growth rate terms of our earlier decomposition of output growth into technological progress and capital deepening.

Methodological Observations on Growth Theory and the Solow Model

Before launching into our first model, a few methodological observations are perhaps useful. Much of macroeconomics is concerned with short-run fluctuations in the macroeconomy. Because consumption accounts for most of GDP, it is natural that much of macroeconomic theory focuses on the dynamics of short-run changes in the savings rate. Short-run fluctuations in employment and unemployment are also a major topic for macroeconomists.

However, these fluctuations are not very important when thinking about the long-run evolution of the economy. For this reason, the models we will consider in this part of

the course will generally make very simple assumptions about the consumption-savings decision and the dynamics of employment. This is not because these topics are unimportant, but rather because macro is not a one-size-fits-all type of field. It would be a daunting task to even attempt to construct a model that explained all interesting macroeconomic phenomena, and any such model would undoubtedly be complicated and unwieldy, making it difficult to learn (and teach). For this reason, macroeconomists tend to adopt a more eclectic approach, with models often being developed with the intention of helping to explain one particular aspect of macroeconomy.

The first model that we will look at in this class, a model of economic growth originally developed by MIT's Robert Solow in the 1950s, is a good example of this general approach. Solow's purpose in developing the model was to take some important aspects of macroeconomics, such as short-run fluctuations in employment and savings rates, as given (i.e. outside the realm of his model to explain) in order to develop a model that shed light on the long-run evolution of the economy. The resulting paper (A Contribution to the Theory of Economic Growth, *QJE*, 1956) remains highly influential even today and, despite its relative simplicity, the model conveys a number of very useful insights about the dynamics of the growth process. Solow is an entertaining writer and the paper is well worth reading. However, I should point out that the way we will discuss the model will follow Chapter 4 of Brad DeLong's textbook more closely than it will Solow's original paper.

The Solow Model's Production Function

The starting point for the Solow model is the assumption that there is a production function with diminishing marginal returns to capital accumulation. This can be represented using a broad range of production functions, but for concreteness, we'll stick with the Cobb-Douglas formulation. In this case, the Solow model assumption implies:

$$Y_t = A_t K_t^\alpha L_t^\beta \quad 0 < \alpha < 1 \quad (19)$$

The assumption that the parameter α is less than one is what generates diminishing marginal returns to capital. In other words, adding extra capital while holding labour input fixed yields ever-smaller increases in output. This can be shown as follows:

$$\frac{\partial Y}{\partial K} = \alpha A_t K_t^{\alpha-1} L_t^\beta \quad (20)$$

$$\left(\frac{\partial^2 Y}{\partial K^2} \right) = \alpha(\alpha - 1) A_t K_t^{\alpha-2} L_t^\beta < 0 \quad (21)$$

This turns out to be the key element of the model because it determines the model's answer to the key question relating to the relative importance of capital deepening and technological progress. In addition to this assumption, applications usually assume that the production function displays constant returns to scale ($\beta = 1 - \alpha$), but for now I have not assumed that here because it is not necessary for deriving the model's main predictions.

Think about why diminishing marginal returns is probably sensible: If a firm acquires an extra unit of capital, it will probably be able to increase its output. But if the firm keeps piling on extra capital without raising the number of workers available to use this capital, the increases in output will probably taper off. In the Cobb-Douglas case, the parameter α dictates the pace of this tapering off. A useful analogy is to the ingredients for a cake: Adding more of a particular ingredient will help to produce more cake, but adding endless amounts of extra flour isn't going to help much unless one adds more of the other ingredients as well. Going further with the analogy, the parameter α gives us an idea of how important an ingredient capital is: The smaller α is the more negative the second derivative becomes, and so the faster it is that diminishing returns sets in.

The Model's Other Ingredients

In addition to the production function, the model has four other equations.

- Capital accumulates according to

$$\dot{K}_t = Y_t - C_t - \delta K_t \quad (22)$$

In other words, the addition to the capital stock each period depends positively on savings (this is a closed-economy model so savings equals investment) and negatively on depreciation, which is assumed to take place at rate δ .

- Labour input grows at rate n :

$$\frac{\dot{L}_t}{L_t} = n \quad (23)$$

- Technological progress occurs at rate g :

$$\frac{\dot{A}_t}{A_t} = g$$

- A fraction s of output is saved each period.

$$Y_t - C_t = sY_t \quad (24)$$

The model does not attempt to explain fluctuations in the rate of population growth, the rate of technological progress, the rate of depreciation of capital or the savings rate. Also, note that I have not put time subscripts on these variables, because we will generally consider these to be constant. However, this does *not* mean that constant values for these parameters is an integral assumption of the Solow model.¹ Indeed, one of the things that we will want to figure out is what happens if these parameters changes. So, for instance, we will be interested in what happens when there is a once-off increase in the savings rate.

Steady-State Growth

The first thing we are going to do with the Solow model is figure out what this economy looks like along a path on which output growth is constant. Macroeconomists refer to such constant growth paths as *steady-state* growth paths. We don't necessarily want to study *only* constant-growth paths, but we will see below that the Solow-model economy tends to converge over time towards such a path.

First note that, given constant growth rates for technology and labour input, all variations in output growth are due to variations in the growth rate of capital input:

$$\frac{\dot{Y}_t}{Y_t} = g + \alpha \frac{\dot{K}_t}{K_t} + \beta n \quad (25)$$

So for output growth to be constant, we must also have capital growth being constant.

We can also show that these growth rates for capital and output must be the same, so that the capital-output ratio is constant along a constant growth. To see this, re-write the capital accumulation equation as

$$\dot{K}_t = sY_t - \delta K_t \quad (26)$$

and divide across by K_t on both sides

$$\frac{\dot{K}_t}{K_t} = s \frac{Y_t}{K_t} - \delta \quad (27)$$

The growth rate of the capital stock depends negatively on the capital-output ratio $\frac{K_t}{Y_t}$. So, for the capital stock to be growing at a constant rate, then $\frac{K_t}{Y_t}$ must be constant. But $\frac{K_t}{Y_t}$ can only be constant if the growth rate of K_t is the same as the growth rate of Y_t .

¹In other words, don't write on the exam that "A weakness of the Solow model is that it assumes the savings rate is constant, which is clearly false."

With this result in mind, we see that the steady-state growth rate must satisfy

$$\frac{\dot{Y}_t}{Y_t} = g + \alpha \frac{\dot{Y}_t}{Y_t} + \beta n \quad (28)$$

Subtracting $\alpha \frac{\dot{Y}_t}{Y_t}$ from both sides, we get

$$(1 - \alpha) \frac{\dot{Y}_t}{Y_t} = g + \beta n \quad (29)$$

So, the steady-state growth rate is

$$\frac{\dot{Y}_t}{Y_t} = \frac{g}{1 - \alpha} + \frac{\beta}{1 - \alpha} n \quad (30)$$

If we have constant returns to scale, so that $\beta = 1 - \alpha$, then we get

$$\frac{\dot{Y}_t}{Y_t} - n = \frac{g}{1 - \alpha} \quad (31)$$

Only the growth rate of technology, g , and the factor controlling the extent of diminishing marginal returns to capital, α , can affect the growth rate of output per worker. This is a key result: All the other parameters have *no effect* on this key steady-state growth rate. For example, economies with higher saving rates do not have faster steady-state growth rates.

Why is this? An increase in the saving rate can raise the growth rate initially by boosting capital accumulation. But diminishing marginal returns implies that during this period capital growth will outstrip output growth: Look at equation (25) and note that because $\alpha < 1$, capital growth does not fully translate into output growth. And this period of capital growing faster than output will not last. Equation (27) tells us that capital growth depends negatively on the capital-output ratio: As capital grows faster than output, capital growth will keep slowing. So higher saving rates can produce temporary increases the growth rate of output, but cannot get the economy to a path involving a faster *steady-state* growth rate. Next, we will formally work through the dynamics of how an economy converges to this steady-state growth rate.

Dynamics of the Capital-Output Ratio

Now recall equation (9) for output per worker. It states that output per worker is a function of A_t and of the capital-output ratio. Because A_t is assumed to grow at a constant rate each period, this means that all of the interesting dynamics for output per worker in this

model stem from the behaviour of the capital-output ratio. We will now describe how this ratio behaves. Using our terminology for the capital-output ratio, we can re-write equation (27) as

$$\frac{\dot{K}_t}{K_t} = \frac{s}{x_t} - \delta \quad (32)$$

Again using logarithm tricks, note that

$$\log x_t = \log \frac{K_t}{Y_t} = \log K_t + \log \frac{1}{Y_t} = \log K_t + \log Y_t^{-1} = \log K_t - \log Y_t \quad (33)$$

Taking derivatives with respect to time we have

$$\frac{\dot{x}_t}{x_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{Y}_t}{Y_t} \quad (34)$$

Now using equation (25) for output growth and equation (32) for capital growth, we can derive a useful equation for the dynamics of the capital-output ratio:

$$\frac{\dot{x}_t}{x_t} = (1 - \alpha) \frac{\dot{K}_t}{K_t} - g - \beta n \quad (35)$$

$$= (1 - \alpha) \left(\frac{s}{x_t} - \frac{g}{1 - \alpha} - \frac{\beta}{1 - \alpha} n - \delta \right) \quad (36)$$

This dynamic equation has a very important property: The growth rate of x_t depends negatively on the value of x_t . In particular, when x_t is over a certain value, it will tend to decline, and when it is under that value it will tend to increase. Thus the capital-output ratio exhibits *convergent dynamics*: It tends to converge to a specific long-run steady-state value. Note the importance of the diminishing marginal productivity of capital feature for this result: If $\alpha = 1$, then the model would not display convergent dynamics.

What is the long-run steady-state value of x_t , which we will label x^* ? It is the value consistent with $\frac{\dot{x}}{x} = 0$. This implies that

$$\frac{s}{x^*} - \frac{g}{1 - \alpha} - \frac{\beta}{1 - \alpha} n - \delta = 0 \quad (37)$$

This solves to give

$$x^* = \frac{s}{\frac{g}{1 - \alpha} + \frac{\beta}{1 - \alpha} n + \delta} \quad (38)$$

Under constant returns, this simplifies to

$$x^* = \frac{s}{\frac{g}{1 - \alpha} + n + \delta} \quad (39)$$

Given this expression for the steady-state capital-output ratio, we can also derive a more intuitive-looking expression to describe the convergence properties of the ratio. To keep the notation simple, we will maintain the constant returns assumption, so that the dynamics of x_t are given by

$$\frac{\dot{x}_t}{x_t} = (1 - \alpha) \left(\frac{s}{x_t} - \frac{g}{1 - \alpha} - n - \delta \right) \quad (40)$$

Multiplying and dividing the right-hand-side of this equation by $(\frac{g}{1 - \alpha} + n + \delta)$:

$$\frac{\dot{x}_t}{x_t} = (1 - \alpha) \left(\frac{g}{1 - \alpha} + n + \delta \right) \left(\frac{s/x_t - \frac{g}{1 - \alpha} - n - \delta}{\frac{g}{1 - \alpha} + n + \delta} \right) \quad (41)$$

The last term inside the brackets can be simplified to give

$$\frac{\dot{x}_t}{x_t} = (1 - \alpha) \left(\frac{g}{1 - \alpha} + n + \delta \right) \left(\frac{1}{x_t} \frac{s}{\frac{g}{1 - \alpha} + n + \delta} - 1 \right) \quad (42)$$

$$= (1 - \alpha) \left(\frac{g}{1 - \alpha} + n + \delta \right) \left(\frac{x^*}{x_t} - 1 \right) \quad (43)$$

$$= (1 - \alpha) \left(\frac{g}{1 - \alpha} + n + \delta \right) \left(\frac{x^* - x_t}{x_t} \right) \quad (44)$$

This equation states that each period the capital-output ratio closes a fraction equal to $\lambda = (1 - \alpha)(\frac{g}{1 - \alpha} + n + \delta)$ of the gap between the current value of the ratio and its steady-state value.

The Steady-State Level of Output Per Worker

We have derived the dynamic behaviour of the capital-output ratio in the Solow model. It turns out to be pretty easy to also derive the model's predictions for the behaviour of output per worker. This is because output per worker is determined by A_t , which we know follows a set path, and by the capital-output ratio, whose dynamics we have just derived.

To see this, apply the take-logs-and-derivatives trick to equation (9) to get the growth rate of output per worker in terms of technological progress and changes in the capital-output ratio:

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{1}{1 - \alpha} \frac{\dot{A}_t}{A_t} + \frac{\alpha}{1 - \alpha} \frac{\dot{x}_t}{x_t} \quad (45)$$

Substituting in the growth rate of x from equation (44) to get

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{g}{1 - \alpha} + \alpha \left(\frac{g}{1 - \alpha} + n + \delta \right) \left(\frac{x^* - x_t}{x_t} \right) \quad (46)$$

Output growth equals the steady-state growth rate $\frac{g}{1-\alpha}$ plus or minus that element due to the capital-output ratio converging towards its steady-state level.

The model also gives us an expression for the steady-state path for output per worker, i.e. the path towards which output is always converging in which $x_t = x^*$. This is obtained by plugging the steady-state capital-output ratio into equation (9) to get

$$\left(\frac{Y_t}{L_t}\right)^* = A_t^{\frac{1}{1-\alpha}} \left(\frac{s}{\frac{g}{1-\alpha} + n + \delta}\right)^{\frac{\alpha}{1-\alpha}} \quad (47)$$

This formula provides a way to calculate the long-run effects on the level of output per worker of changes in the savings rate, depreciation rate etc.

Lessons from the Solow Model

A number of lessons can be drawn from the Solow model:

- It helps to settle the “technological progress versus capital deepening” question decisively in favour of technological progress. In the long-run, growth is dependent on sustaining improvements in technological efficiency. We have derived this result using a Cobb-Douglas production function, but in fact this result holds for any production function featuring diminishing marginal productivity for capital.
- The model serves as a useful warning against basing policy recommendations on identities rather than fully-worked out economic models. Equation (18) is an identity that says that growth is a function of both capital deepening and technological progress and empirical decompositions of this type (known as *growth accounting* studies) are quite commonly carried out, with researchers concluding that a certain fraction of the growth in output per worker over a certain period was due to capital deepening. However, the model points out that, in the long-run, one cannot sustain capital deepening without technological progress. Ultimately, it is technological progress that offsets the effects of diminishing marginal returns, and thus allows capital deepening to play a role along the steady growth path.
- While the model predicts that changes in the saving-investment rate do not increase the rate of growth in the long-run, it does make very precise predictions about exactly how much such a change will increase the *level* of output, as well as the speed with which the economy will converge towards this new higher level.

Concrete Example 1: Convergence Dynamics

Often, the best way to understand dynamic models is to load them onto the computer and see them run. This is easily done using spreadsheet software such as Excel or econometrics-oriented packages such as RATS. Figures 1 to 3 provide examples of the behaviour over time of two economies, one that starts with a capital-output ratio that is half the steady-state level, and other that starts with a capital output ratio that is 1.5 times the steady-state level.

The parameters chosen were $s = 0.2$, $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, $g = 0.02$, $n = 0.01$, $\delta = 0.06$. Together these parameters are consistent with a steady-state capital-output ratio of 2. To see, this plug these values into (39):

$$\left(\frac{K}{Y}\right)^* = \frac{s}{\frac{g}{1-\alpha} + n + \delta} = \frac{0.2}{1.5 * 0.02 + 0.01 + 0.06} = 2 \quad (48)$$

The first chart shows how the two capital-output ratios converge, somewhat slowly, over time to their steady-state level. This slow convergence is dictated by our choice of parameters: Our “convergence speed” is:

$$\lambda = (1 - \alpha)\left(\frac{g}{1 - \alpha} + n + \delta\right) = \frac{2}{3}(1.5 * 0.02 + 0.01 + 0.06) = 0.067 \quad (49)$$

So, the capital-output ratio converges to its steady-state level at a rate of about 7 percent per period. These are fairly standard parameter values for annual data, so this should be understood to mean 7 percent per year.

The second chart shows how output per worker evolves over time in these two economies. Both economies exhibit growth, but the capital-poor economy grows faster during the convergence period than the capital-rich economy. These output per worker differentials may seem a little small on this chart, but the final chart shows the behaviour of the growth rates, and this chart makes it clear that the convergence dynamics can produce substantially different growth rates depending on whether an economy is above or below its steady-state capital-output ratio. During the initial transition periods, the capital-poor economy grows at rates over 6 percent, while the capital-rich economy grows at under 2 percent. Over time, both economies converge towards the steady-state growth rate of 3 percent.

Concrete Example 2: Changes in Parameters

Figures 4 to 6 examine what happens when the economy is moving along the steady-state path consistent with the parameters just given, and then one of the parameters is changed. Specifically, it examines the effects of changes in s , δ and g .

Consider first an increase in the savings rate to $s = 0.25$. This has no effect on the steady-state growth rate. But it does change the steady-state capital-output ratio from 2 to 2.5. So the economy now finds itself with too little capital relative to its new steady-state capital-output ratio. The growth rate jumps immediately and only slowly returns to the long-run 3 percent value. The faster pace of investment during this period gradually brings the capital-output ratio into line with its new steady-state level.

The increase in the savings rate permanently raises the level of output per worker relative to the path that would have occurred without the change. However, for our parameter values, this effect is not that big. This is because the long-run effect of the savings rate on output per worker is determined by $s^{\frac{\alpha}{1-\alpha}}$, which in this case is $s^{0.5}$. So in our case, 25 percent increase in the savings rate produces an 11.8 percent increase in output per worker ($1.25^{0.5} = 1.118$). More generally, a doubling of the savings rate raises output per worker by 41 percent ($2^{0.5} = 1.41$).

The charts also show the effect of an increase in the depreciation rate to $\delta = 0.11$. This reduces the steady-state capital-output ratio to 4/3 and the effects of this change are basically the opposite of the effects of the increase in the savings rate.

Finally, there is the increase in the rate of technological progress. I've shown the effects of a change from $g = 0.02$ to $g = 0.03$. This increases the steady-state growth rate of output per worker to 0.045. However, as the charts show there is another effect: A faster steady-state growth rate for output reduces the steady-state capital-output ratio. Why? The increase in g raises the long-run growth rate of output; this means that each period the economy needs to accumulate more capital than before just to keep the capital-output ratio constant. Again, without a change in the savings rate that causes this to happen, the capital-output ratio will decline. So, the increase in g means that—as in the depreciation rate example—the economy starts out in period 25 with too much capital relative to its new steady-state capital-output ratio. For this reason, the economy doesn't jump straight to its new 4.5 percent growth rate of output per worker. Instead, after an initial jump in the growth rate, there is a very gradual transition the rest of the way to the 4.5 percent growth rate.

A Real-World Example: Europe versus the US

If one is willing to make an assumption about the value of α , one can use data on output, labor input, and capital input to calculate TFP growth consistent with the equation

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} + \alpha \frac{\dot{K}_t}{K_t} + (1 - \alpha) \frac{\dot{L}_t}{L_t} \quad (50)$$

One way to come up with a value of α is to note that if under certain conditions (cost minimisation and perfect factor markets) $1 - \alpha$ can be equated with the share of income paid to labor, which is a series that can be calculated from national income data. Such calculations usually point to a value of about one-third. In recent research I carried out with my colleague Kieran McQuinn, we calculated TFP growth for the US and for the Euro Area economy based on $\alpha = \frac{1}{3}$.² A table summarising some the results is attached. The table shows output growth (Δy), TFP growth (Δa), capital growth (Δk), and growth in labour input as measured by hours worked (Δl).

A key result in this table is that the Euro area used to have faster TFP growth than the US (3.0 percent per year in the 1970s compared with 1.1 percent) but more recently has had significantly slower TFP growth (0.4 percent per year since 2000, compared with 1.5 percent in the US). The paper uses the Solow model to discuss the long-run implications of the persistence of such slow rates of TFP growth. The period since 2000 has seen disappointing growth in the Euro area: Output per worker growth has been only 1.0 percent per year, compared with 1.8 percent in the 1990s and 2.4 percent in the 1980s. However, the Solow model points to even lower growth in the future if the current sluggish pace of TFP growth is maintained. For instance, an economy with a constant investment rate, constant growth rate of labor input, and TFP growth of $g = 0.004$ per year will tend to converge to a long run growth rate of $\frac{g}{1-\alpha} = \frac{0.004}{1-\frac{2}{3}} = 0.006$. Our paper describes this scenario of slow convergence to a steady-state path of six-tenths of percentage point per year growth in output per worker. It also discusses the potential effects on growth of various types of policy initiatives.

²Kieran McQuinn and Karl Whelan, "Prospects for Growth in the Euro Area" Available at www.karlwhelan.com

Table 1: Decomposition of Euro Area and US Output Growth Rates (%)

Period	Euro Area				United States			
	Δy	Δa	Δk	Δl	Δy	Δa	Δk	Δl
1970:1-2006:2	2.4	1.7	0.8	-0.0	3.1	1.3	0.9	0.9
1970:1-1980:1	3.5	3.0	1.0	-0.5	3.3	1.1	1.0	1.2
1980:1-1990:1	2.2	1.7	0.6	-0.1	3.1	1.1	0.9	1.1
1990:1-2000:1	2.1	1.2	0.7	0.2	3.1	1.4	0.8	0.9
2000:1-2006:2	1.6	0.4	0.8	0.4	2.6	1.5	0.8	0.3
1996:1-2006:2	2.1	0.8	0.7	0.6	3.3	1.6	0.9	0.8
1996:1-2001:1	3.0	1.3	0.7	0.9	3.8	1.6	1.0	1.2
2001:1-2006:2	1.4	0.3	0.7	0.4	2.7	1.6	0.8	0.4

Figure 1

Convergence Dynamics for the Capital-Output Ratio

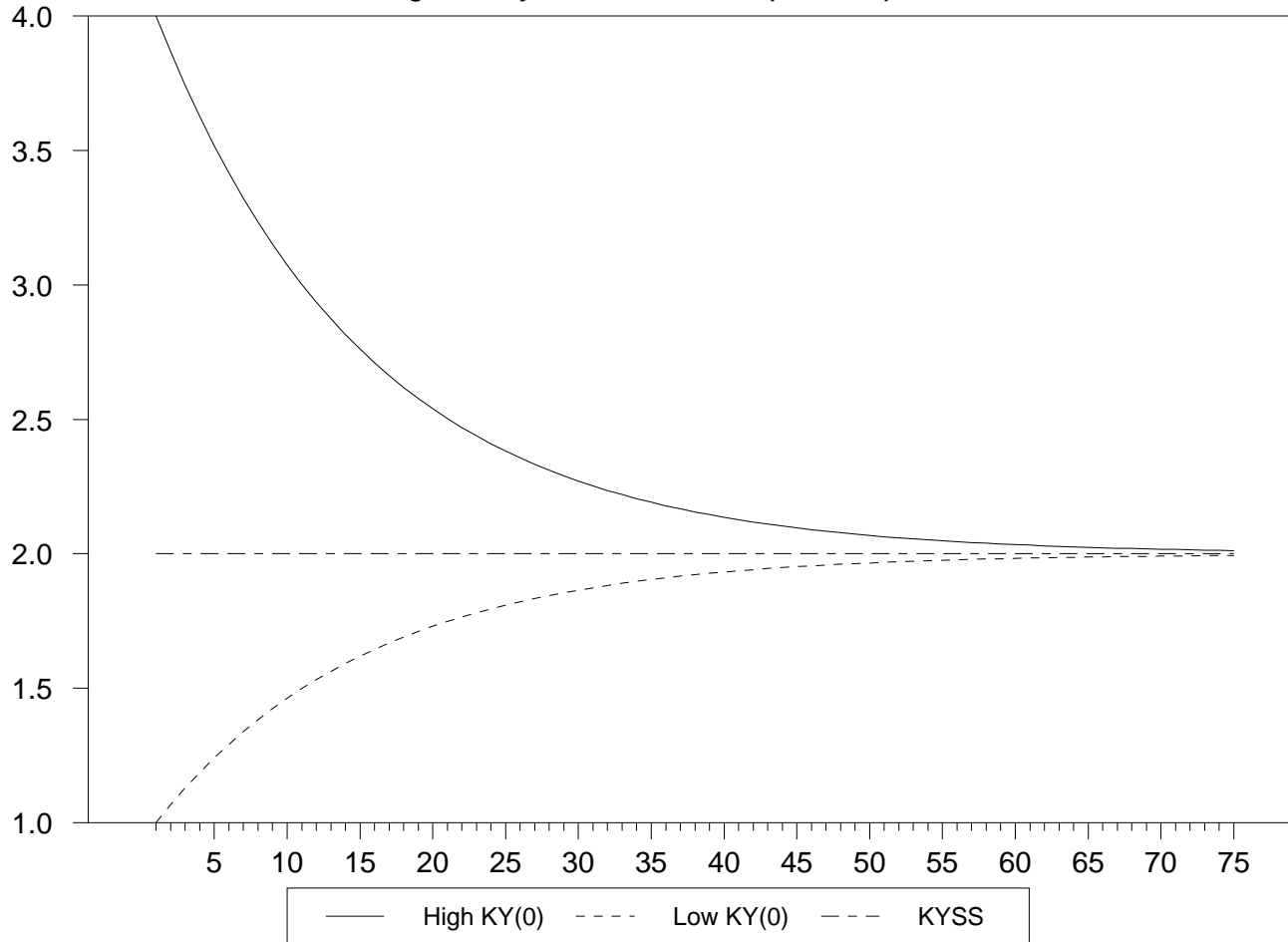


Figure 2

Convergence Dynamics for Output Per Worker

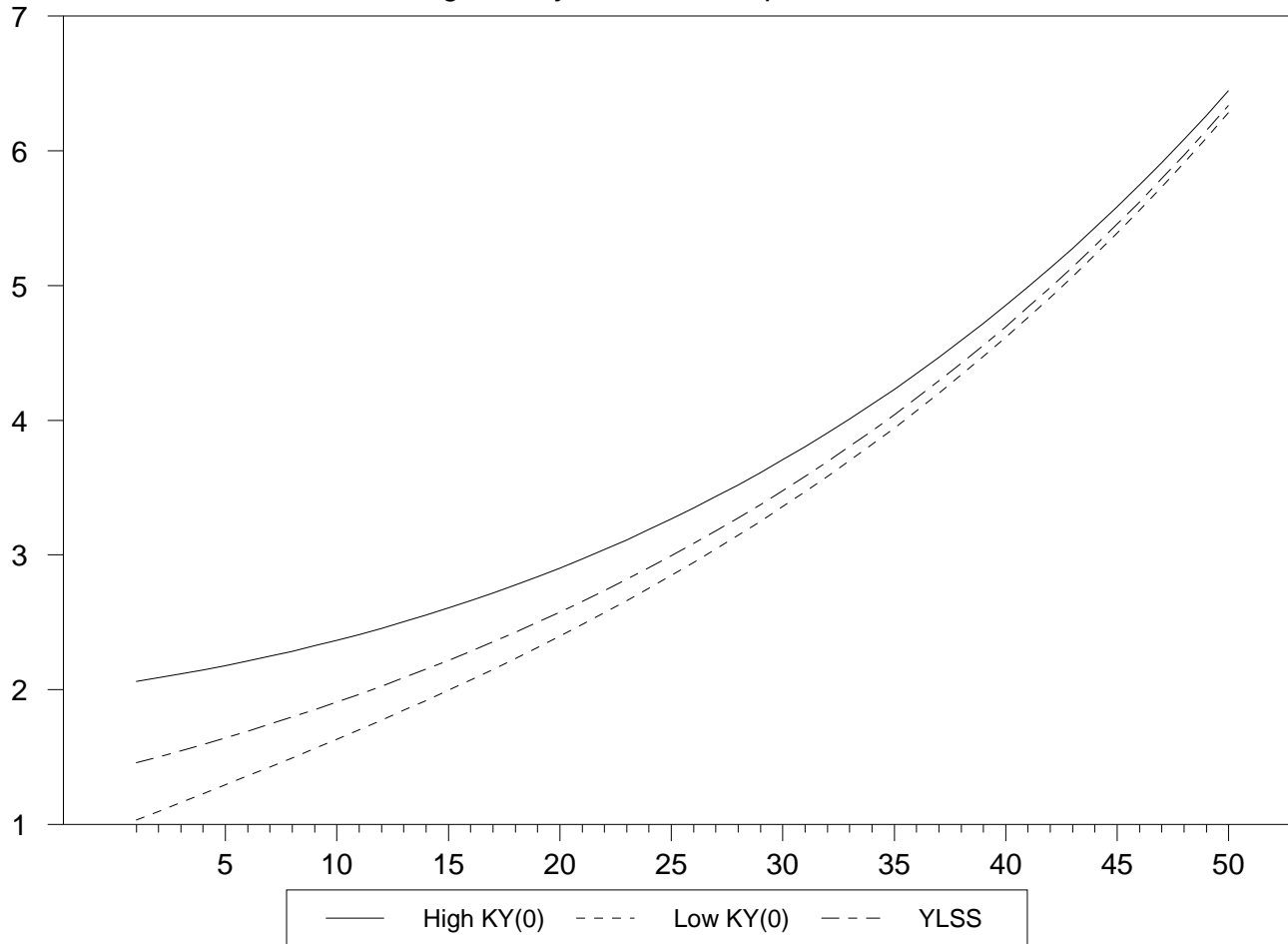


Figure 3

Convergence Dynamics for Growth Rates of Output Per Worker

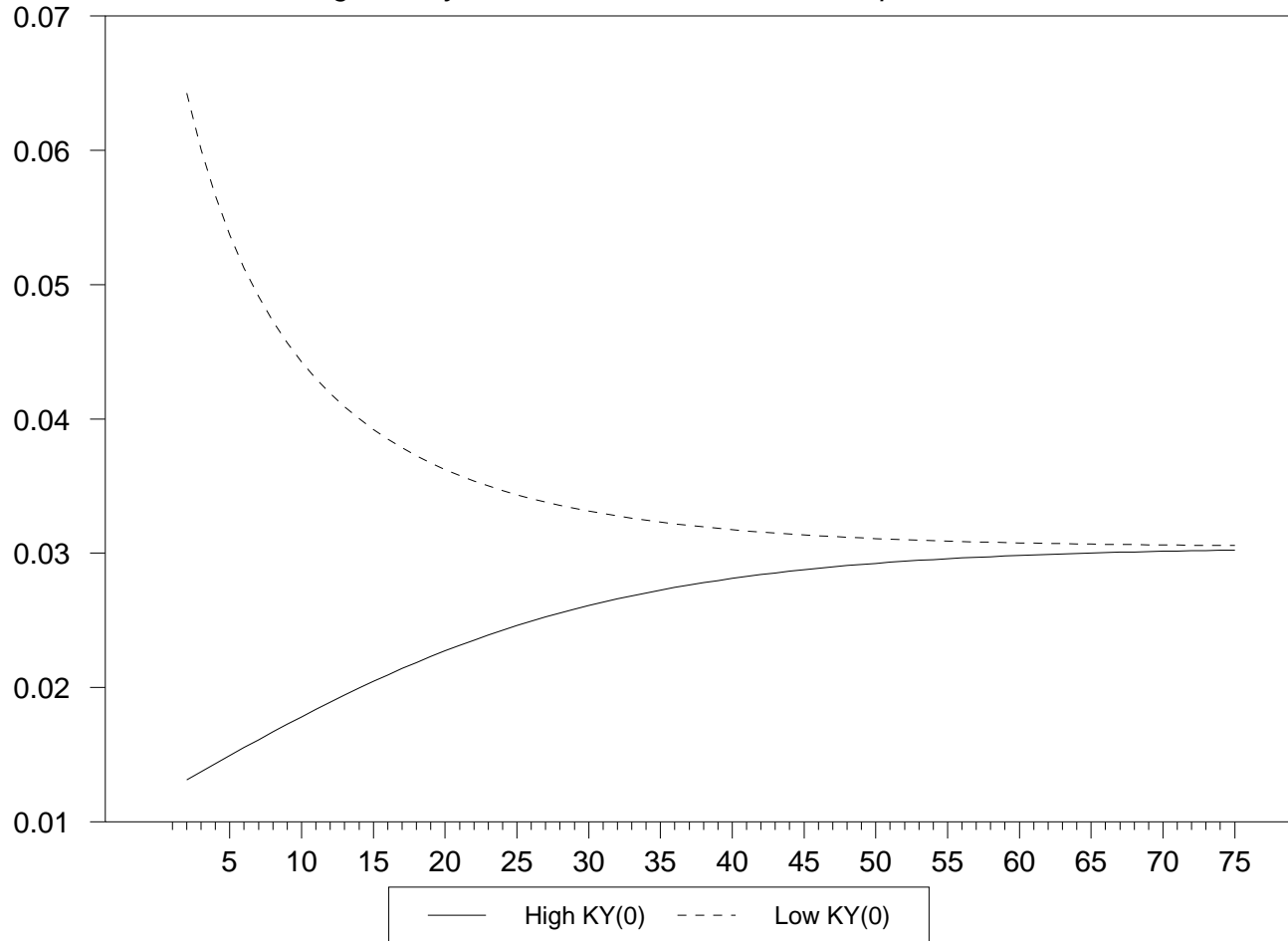


Figure 4

Capital-Output Ratios: Effects of Increases in

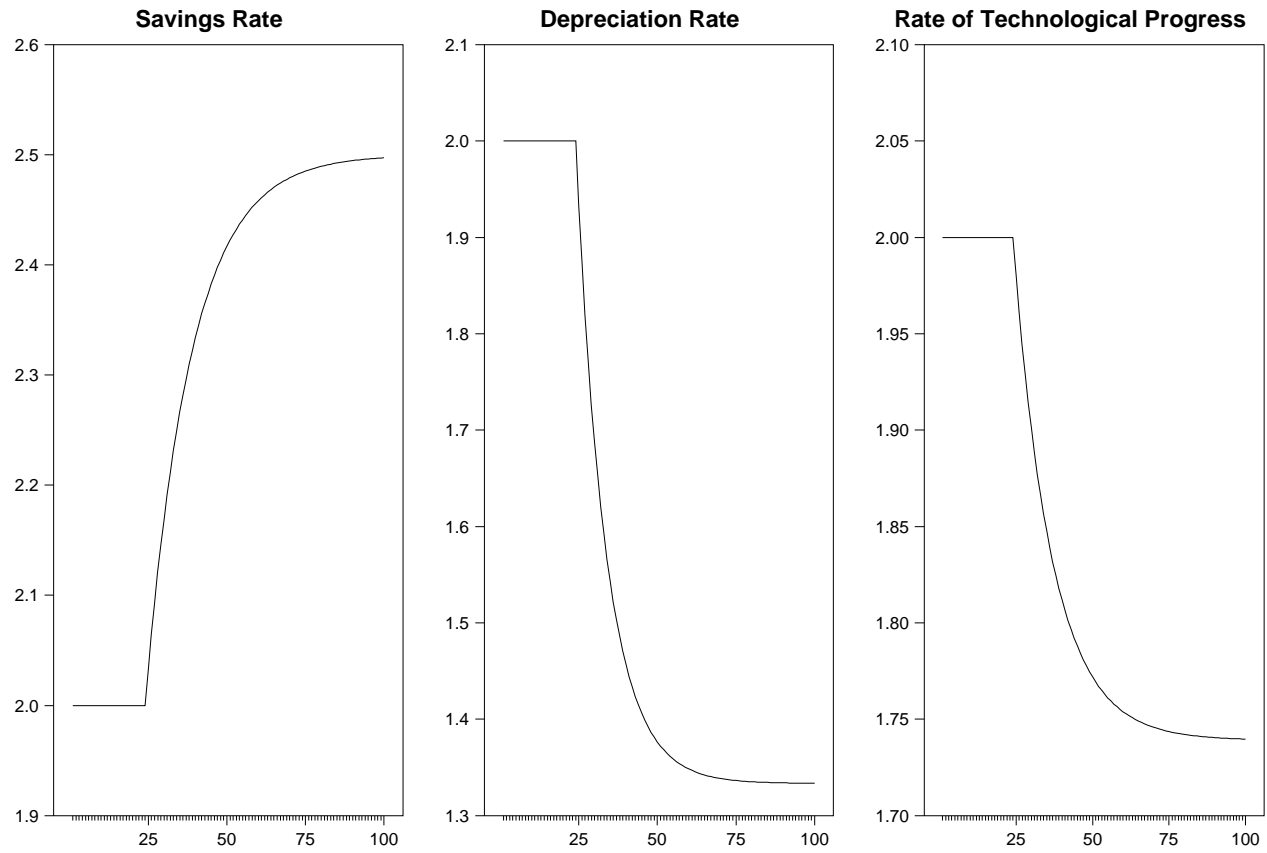


Figure 5

Growth Rates of Output Per Hour: Effects of Increases in

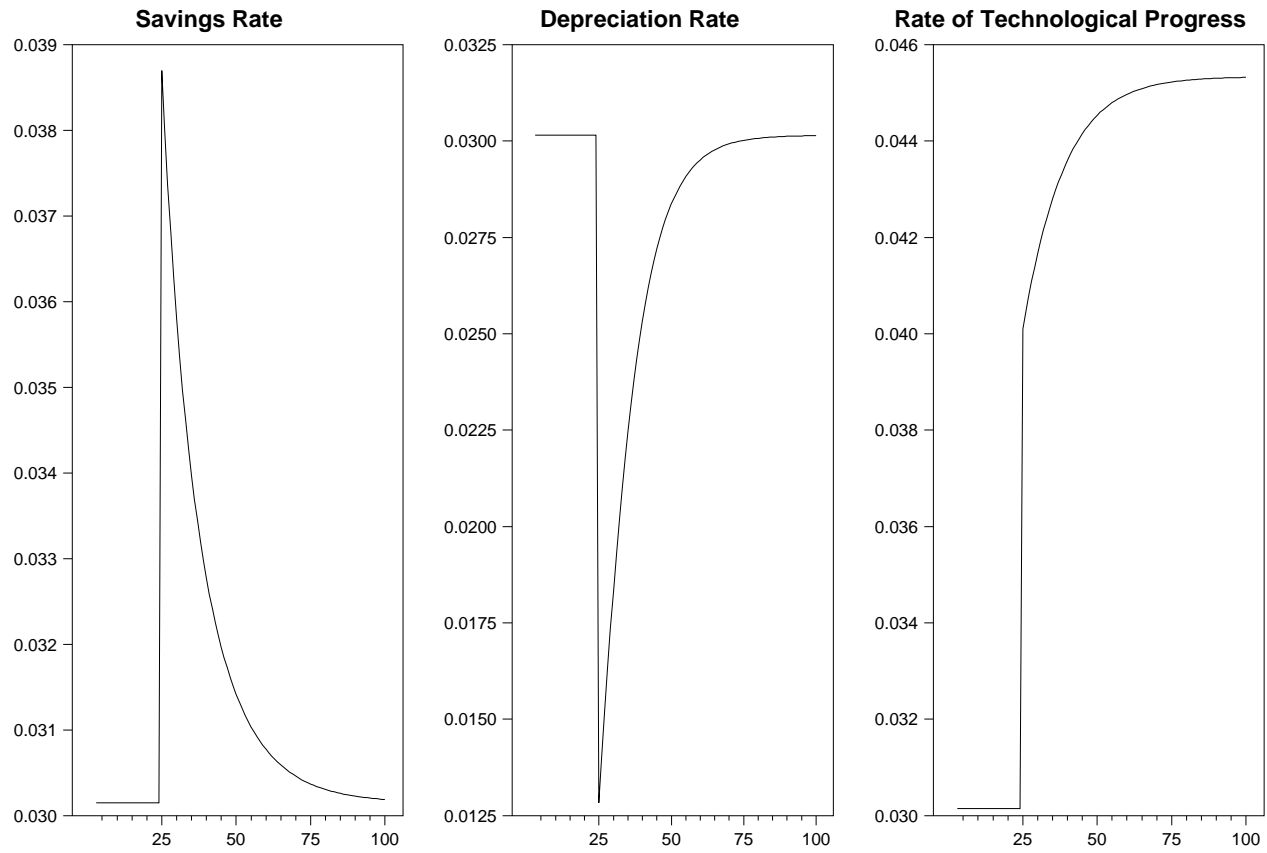


Figure 6

Output Per Hour: Effects of Increases in

