MA Advanced Macroeconomics:
7. The Real Business Cycle Model

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Working Through A DSGE Model

- We have described methods for solving and simulating linear models with lags, leads and rational expectations.
- Now it is time to go through a particular model to see how these methods get combined with economic theory.
- Specifically, we will work through a version of the Real Business Cycle (RBC) model—introduced in a famous 1982 paper by Finn Kydland and Edward Prescott—is the original DSGE model.[1]
- We will set out a basic RBC model and discuss how the model's first-order conditions can be turned into a system of linear difference equations of the form we know how to solve.
- This will require explaining another new technique, known as *log-linearization*.
- While many now question the specific assumptions underlying the early RBC models, the *methodology* has endured.

[1] "Time to Build and Aggregate Fluctuations," *Econometrica*, November 1982, Volume 50, pages 1345-1370. This paper was cited in the 2004 Nobel prize award given to Kydland and Prescott.

Part I

Introduction to the RBC Model

An RBC Model

- The basic RBC model assume perfectly functioning competitive markets, so the outcomes generated by decentralized decisions by firms and households can be replicated as the solution to a social planner problem.
- The social planner wants to maximize

$$E_t \left[\sum_{i=0}^{\infty} \beta^i \left(U(C_{t+i}) - V(N_{t+i}) \right) \right]$$

where C_t is consumption, N_t is hours worked, and β is the representative household's rate of time preference.

- The economy faces constraints described by

$$\begin{aligned} Y_t &= C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\ K_t &= I_t + (1-\delta) K_{t-1} \end{aligned}$$

and a process for the technology term A_t, usually a log-linear AR(1):

$$\log A_t = (1-\rho) \log A^* + \rho \log A_{t-1} + \epsilon_t$$

Discussion of the RBC Approach: 1

- The RBC approach is often criticised but it is worth understanding the reasons why it is a useful baseline model.
- *Criticism: Perfect Markets and Rational Expectations:*
  - Can the economy really be characterized as a perfectly competitive market equilibrium solution describing the behaviour of a set of completely optimizing rational agents? Surely we know that markets are not always competitive and surely people are not always completely rational in their economic decisions?
- The RBC model should be seen as a *benchmark* against which more complicated models can be assessed. If a model with optimizing agents and instantaneous market clearing can explain the business cycle, then can market imperfections such as sticky prices really be seen as crucial to understanding macroeconomic fluctuations?
- Imperfect competition means the decentralized market outcome cannot be characterized as the outcome of a social planner problem. So, one needs to derive the FOCs from separate modelling of the decisions of firms and households. But this is easily done.

Discussion of the RBC Approach: 2

- *Criticism: Monetary and Fiscal Policy*:
  - RBC models exhibit complete monetary neutrality, so there is no role at all for monetary policy, something which many people think is crucial to understanding the macroeconomy. We haven't put government spending in this model, but if we did, the model would exhibit Ricardian equivalence, so the effects of fiscal policy would be different from what most people imagine them to be.
- However, most modern models that build on the RBC approach have introduced mechanisms that allow monetary and fiscal policy to have Keynesian effects. For instance, most of the DSGE models used these days feature various forms of sticky prices and sticky wages, which lead to real effects for monetary policy.

Discussion of the RBC Approach: 3

- *Criticism: Skepticism about Technology Shocks*:
  - RBC models give primacy to technology shocks as the source of economic fluctuations (all variables apart from A_t are deterministic). But what are these shocks?
  - Is it really credible that all economic fluctuations, including recessions, are just an optimal response to technology shocks? If we were experiencing these big shocks wouldn't we be reading about them in the newspapers? Wouldn't recessions have to be periods in which their is outright technological regress, so that (for some reason) firms are using less efficient technologies than previously—is this credible?
- Technology shocks are probably more important than one might think at first. Growth theory teaches us that increases in TFP are the ultimate source of long-run growth in output per hour. Is there any particular reason to think that these increases have to take place in a steady trend-like manner? Put this way, random fluctuations in TFP growth doesn't seem so strange. Also, at least in theory, RBC models could generate recessions without outright declines in technology. This can happen if the elasticity of output with respect to technology is greater than one.

Part II

Solving the RBC Model

A Technical Note

- Our problem involves maximizing
$$E_t\left[\sum_{i=0}^{\infty}\beta^i\left(U(C_{t+i})-V(N_{t+i})\right)\right]$$
where the uncertainty here relates to not knowing future values of A_t.
- Technically, the best way to solve these problems is using *stochastic dynamic programming* but I don't have time to teach that. Instead, I will effectively treat it as a deterministic problem and then substitute E_tX_{t+i} for X_{t+i}.
- Justification? Suppose
$$G(x)=\sum_{k=1}^{N}p_kF(a_k,x)$$
- This is maximized by setting
$$G'(x)=\sum_{k=1}^{N}p_kF'(a_k,x)=E_tF'(x)=0$$
So, the FOCs for for maximizing $E_tF(x)$ are just $E_tF'(x)=0$.

Formulating the Social Planner's Problem

- Remember our two constraints

$$\begin{aligned} Y_t &= C_t + I_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\ K_t &= I_t + (1-\delta) K_{t-1} \end{aligned}$$

- We can simplify the problem by combining them into one equation:

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\delta) K_{t-1}$$

- We can then formulate the social planner's problem as a Lagrangian problem involving picking a series of values for consumption and labour, subject to satisfying a series of constraints of the form just described:

$$\begin{aligned} L &= E_t \sum_{i=0}^{\infty} \beta^i \left[U(C_{t+i}) - V(N_{t+i}) \right] \\ &\quad + E_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} \left[A_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{1-\alpha} + (1-\delta) K_{t+i-1} - C_{t+i} - K_{t+i} \right] \end{aligned}$$

How to Get the First-Order Conditions

- This equation might look a bit intimidating. It involves an infinite sum, so technically there is an infinite number of first-order conditions for current and expected future values of C_t, K_t and N_t.
- But the problem is less hard than this makes it sound, Note that the time-t variables appear in this sum as

$$U(C_t) - V(N_t) + \lambda_t \left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\delta) K_{t-1} \right)$$
$$+ \beta E_t \left[\lambda_{t+1} \left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta) K_t \right) \right]$$

- After that, the time-t variables don't ever appear again. So, the FOCs for the time-t variables consist of differentiating this equation with respect to these variables and setting the derivatives equal to zero.
- Then, the time $t+n$ variables appear exactly as the time t variables do, except that they are in expectation form and they are multiplied by the discount rate β^n. But this means the FOCs for the time $t+n$ variables will be identical to those for the time t variables. So differentiating this equation gives us the equations for the optimal dynamics at all times.

The First-Order Conditions

- Differentiating

$$U(C_t) - V(N_t) + \lambda_t \left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\delta) K_{t-1}\right)$$
$$+\beta E_t \left[\lambda_{t+1} \left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta) K_t\right)\right]$$

- We get following first-order conditions:

$$\frac{\partial L}{\partial C_t} \quad : \quad U'(C_t) - \lambda_t = 0$$
$$\frac{\partial L}{\partial K_t} \quad : \quad -\lambda_t + \beta E_t \left[\lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta\right)\right] = 0$$
$$\frac{\partial L}{\partial N_t} \quad : \quad -V'(N_t) + (1-\alpha) \lambda_t \frac{Y_t}{N_t} = 0$$
$$\frac{\partial L}{\partial \lambda_t} \quad : \quad A_t K_{t-1}^{\alpha} N_t^{1-\alpha} - C_t - K_t + (1-\delta) K_{t-1} = 0$$

The Keynes-Ramsey Condition

- Define the marginal value of an additional unit of capital next year as

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_t} + 1 - \delta$$

- Then the FOC for capital can be written as

$$\lambda_t = \beta E_t \left(\lambda_{t+1} R_{t+1} \right)$$

- This can then be combined with the FOC for consumption to give

$$U'(C_t) = \beta E_t \left[U'(C_{t+1}) R_{t+1} \right]$$

- Interpretation:
  - Decrease consumption by Δ today, at a loss of $U'(C_t)\Delta$ in utility.
  - Invest to get $R_{t+1}\Delta$ tomorrow.
  - Worth $\beta E_t \left[U'(C_{t+1}) R_{t+1} \Delta \right]$ in terms of today's utility.
  - Along an optimal path, must be indifferent.

CRRA Consumption and Separable Consumption-Leisure

- We are going to work with a utility function of the form:

$$U(C_t) - V(N_t) = \frac{C_t^{1-\eta}}{1-\eta} - aN_t$$

- This formulation of the Constant Relative Risk Aversion (CRRA) utility from consumption and separate disutility from labour turns out to be necessary for the model to have a stable growth path solution.
- The Keynes-Ramsey condition becomes

$$C_t^{-\eta} = \beta E_t \left(C_{t+1}^{-\eta} R_{t+1} \right)$$

- And the condition for optimal hours worked becomes

$$-a + (1-\alpha) C_t^{-\eta} \frac{Y_t}{N_t} = 0$$

The Full Set of Model Equations

- The RBC model can then be defined by the following six equations (three identities describing resource constraints, one a definition, and two FOCs describing optimal behaviour)

$$
\begin{aligned}
Y_t &= C_t + I_t \\
Y_t &= A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\
K_t &= I_t + (1-\delta) K_{t-1} \\
R_t &= \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta \\
C_t^{-\eta} &= \beta E_t \left(C_{t+1}^{-\eta} R_{t+1} \right) \\
\frac{Y_t}{N_t} &= \frac{a}{1-\alpha} C_t^{\eta}
\end{aligned}
$$

  and the process for the technology variable

$$
\log A_t = (1-\rho) \log A^* + \rho \log A_{t-1} + \epsilon_t
$$

- These are not a set of linear difference equations, but a mix of both linear and nonlinear equations: Haven't I been saying that DSGE modelling is all about sets of linear difference equations?

Part III

Log-Linearization

Linearization

- In general, nonlinear systems like this cannot be solved analytically. However, it turns out their solution can be very well approximated by a corresponding set of linear equations.
- The idea is to use Taylor series approximations. In general, any nonlinear function $F(x_t, y_t)$ can be approximated around any point (x_t^*, y_t^*) using the formula

$$\begin{aligned} F(x_t, y_t) &= F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) \\ &+ F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t - x_t^*)(y_t - y_t^*) \\ &+ F_{yy}(x_t^*, y_t^*)(y_t - y_t^*)^2 + ... \end{aligned}$$

- If the gap between (x_t, y_t) and (x_t^*, y_t^*) is small, then terms in second and higher powers and cross-terms will all be very small and can be ignored, leaving something like

$$F(x_t, y_t) \approx \alpha + \beta_1 x_t + \beta_2 y_t$$

- But if we "linearize" around a point that (x_t, y_t) is far away from, then this approximation will not be accurate.

Log-Linearization

- DSGE models use a particular version of this technique. They take logs and then linearize the logs of variables around a simple "steady-state" path in which all real variables are growing at the same rate.
- The steady-state path is relevant because the stochastic economy will, on average, tend to fluctuate around the values given by this path, making the approximation an accurate one.
- This gives us a set of linear equations in the deviations of the logs of these variables from their steady-state values.
- Remember that log-differences are approximately percentage deviations

$$\log X - \log Y \approx \frac{X - Y}{Y}$$

so this approach gives us a system that expresses variables in terms of their percentage deviations from the steady-state paths. It can be thought of as giving a system of variables that represents the business-cycle component of the model. Coefficients are elasticities and IRFs are easy to interpret.
- Also log-linearization is easy. It doesn't require taking lots of derivatives.

How Log-Linearization Works

- We will use lower-case letters to define log-deviations of variables from their steady-state values.

$$x_t = \log X_t - \log X^*$$

- The key to the log-linearization method is that every variable can be written as

$$X_t = X^* \frac{X_t}{X^*} = X^* e^{x_t}$$

- The big trick is that a first-order Taylor approximation of e^{x_t} is given by

$$e^{x_t} \approx 1 + x_t$$

- So, we can write variables as

$$X_t \approx X^* (1 + x_t)$$

- The second trick is for variables multiplying each other such as

$$X_t Y_t \approx X^* Y^* (1 + x_t)(1 + y_t) \approx X^* Y^* (1 + x_t + y_t)$$

  i.e. you set terms like $x_t y_t = 0$ because we are looking at small deviations from steady-state and multiplying these small deviations together one gets a term close to zero.

Anything Else?

- No, that's it.
- Substitute these approximations for the variables in the model, lots of terms end up canceling out, and when you're done you've got a system in the deviations of logged variables from their steady-state values.
- The paper on the reading list by Harald Uhlig discusses this stuff in a bit more detail and provides more examples.
- But the best way to understand this stuff is to see it at work, so let's work through some examples from the RBC model.
- Note that we have assumed that technology (the source of all long-run growth in this economy) is given by

$$a_t = \rho a_{t-1} + \epsilon_t$$

  so there is no trend growth in this economy.
- This means that the steady-state variables are all constants. Technically, there is no great difficulty in modelling an economy with trend growth but this case is a bit simpler.

Log-Linearization Example 1

- Start with

$$Y_t = C_t + I_t$$

- Re-write it as

$$Y^* e^{y_t} = C^* e^{c_t} + I^* e^{i_t}$$

- Using the first-order approximation, this becomes

$$Y^* (1 + y_t) = C^* (1 + c_t) + I^* (1 + i_t)$$

- Note, though, that the steady-state terms must obey identities so

$$Y^* = C^* + I^*$$

- Canceling these terms on both sides, we get

$$Y^* y_t = C^* c_t + I^* i_t$$

  which we will write as

$$y_t = \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t$$

Log-Linearization Example 2

- Now consider

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$$

- This can be re-written in terms of steady-states and log-deviations as

$$Y^* e^{y_t} = \left(A^* e^{a_t}\right)\left(K^*\right)^{\alpha} e^{\alpha k_{t-1}} \left(N^*\right)^{1-\alpha} e^{(1-\alpha)n_t}$$

- Again, use the fact the steady-state values obey identities so that

$$Y^* = A^* \left(K^*\right)^{\alpha} \left(N^*\right)^{1-\alpha}$$

- So canceling gives

$$e^{y_t} = e^{a_t} e^{\alpha k_{t-1}} e^{(1-\alpha)n_t}$$

- Using first-order Taylor approximations, this becomes

$$(1 + y_t) = (1 + a_t)(1 + \alpha k_{t-1})(1 + (1-\alpha) n_t)$$

- Ignoring cross-products of the log-deviations, this simplifies to

$$y_t = a_t + \alpha k_{t-1} + (1-\alpha) n_t$$

The Full Log-Linearized System

Once all the equations have been log-linearized, we have a system of seven equations of the form

$$\begin{aligned}
y_t &= \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t \\
y_t &= a_t + \alpha k_{t-1} + (1-\alpha) n_t \\
k_t &= \frac{I^*}{K^*} i_t + (1-\delta) k_{t-1} \\
n_t &= y_t - \eta c_t \\
c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
r_t &= \left(\frac{\alpha}{R^*} \frac{Y^*}{K^*} \right) (y_t - k_{t-1}) \\
a_t &= \rho a_{t-1} + \epsilon_t
\end{aligned}$$

We are nearly ready to put the model on the computer. However, notice that three of the equations have coefficients that are values relating to the steady-state path. These need to be calculated.

Part IV

Calculating the Steady-State

The Steady-State Interest Rate

- We need to calculate $\frac{C^*}{Y^*}$, $\frac{I^*}{K^*}$ and $\frac{\alpha}{R^*}\frac{Y^*}{K^*}$
- We do this by taking the original non-linearized RBC system and figuring out what things look like along a zero growth path.
- Start with the steady-state interest rate. This is linked to consumption behaviour via the so-called Euler equation (or Keynes-Ramsey condition):

$$1 = \beta E_t \left(\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right)$$

- Because we have no trend growth in technology in our model, the steady-state features consumption, investment, and output all taking on constant values with no uncertainty.
- Thus, in steady-state, we have $C_t^* = C_{t+1}^* = C^*$, so

$$R^* = \beta^{-1}$$

In a no-growth economy, the rate of return on capital is determined by the rate of time preference.

Other Steady-State Calculations

- Take the equation for the rate of return on capital

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta$$

- In steady-state, we have

$$R^* = \beta^{-1} = \alpha \frac{Y^*}{K^*} + 1 - \delta$$

- So, in steady-state, we have

$$\frac{Y^*}{K^*} = \frac{\beta^{-1} + \delta - 1}{\alpha}$$

- Together with the steady-state interest equation, this tells us that

$$\frac{\alpha}{R^*} \frac{Y^*}{K^*} = \alpha\beta \left(\frac{\beta^{-1} + \delta - 1}{\alpha} \right) = 1 - \beta \left(1 - \delta\right)$$

which is the one of the steady-state values required

Investment-Capital and Investment-Output Ratios

- Next, we use the identity

$$K_t = I_t + (1 - \delta) K_{t-1}$$

- And the fact that in steady-state we have $K_t^* = K_{t-1}^* = K^*$, to give

$$\frac{I^*}{K^*} = \delta$$

  which was also required.

- This can then be combined with the previous steady-state ratio to give

$$\frac{I^*}{Y^*} = \frac{\frac{I^*}{K^*}}{\frac{Y^*}{K^*}} = \frac{\alpha\delta}{\beta^{-1} + \delta - 1}$$

- And obviously

$$\frac{C^*}{Y^*} = 1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1}$$

  which gives us the other required steady-state ratios.

The Final System

Using these steady-state identities, our system becomes

$$
\begin{aligned}
y_t &= \left(1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1}\right) c_t + \left(\frac{\alpha\delta}{\beta^{-1} + \delta - 1}\right) i_t \\
y_t &= a_t + \alpha k_{t-1} + (1 - \alpha)\, n_t \\
k_t &= \delta i_t + (1 - \delta)\, k_{t-1} \\
n_t &= y_t - \eta c_t \\
c_t &= E_t c_{t+1} - \frac{1}{\eta} E_t r_{t+1} \\
r_t &= (1 - \beta\,(1 - \delta))\,(y_t - k_{t-1}) \\
a_t &= \rho a_{t-1} + \epsilon_t
\end{aligned}
$$

This is written in the standard format for systems of linear stochastic difference equations. So, once we make assumptions about the underlying parameter values $(\alpha, \beta, \delta, \eta, \rho)$ we can apply solution algorithms such as the Binder-Pesaran program to obtain a reduced-form solution, and thus simulate the model on the computer.

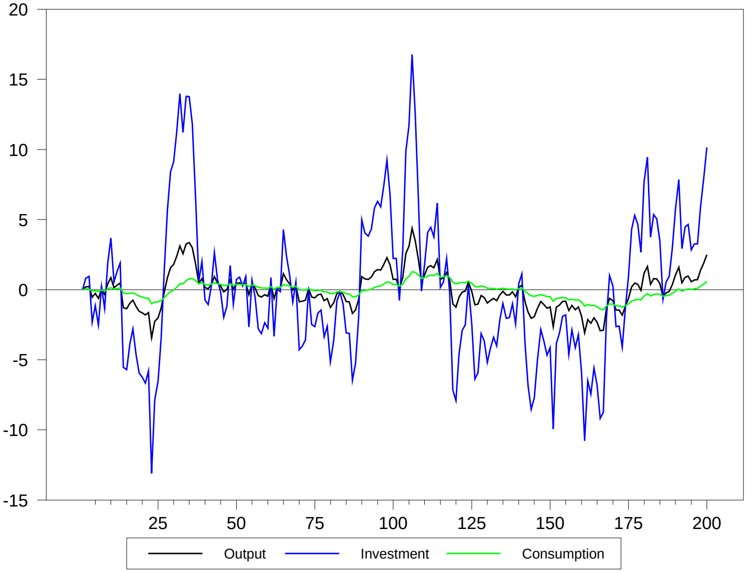
Part V

Simulating the Model

Parameterizing, Simulating and Checking IRFs

- The next few pages show some charts the illustrate the properties of this model.
- The uses parameter values intended for analysis of quarterly time series: $\alpha = \frac{1}{3}$, $\beta = 0.99$, $\delta = 0.015$, $\rho = 0.95$, and $\eta = 1$.
- The first chart shows results from a 200-period simulation of this model. It demonstrates the main successful feature of the RBC model: It generates actual business cycles and they don't look too unrealistic.
- In particular, reasonable parameterizations of the model can roughly match the magnitude of observed fluctuations in output, and the model can match the fact that investment is far more volatile than consumption.
- In the early days of RBC research, this ability to match business cycle dynamics was considered a major strength, and many economists began to claim that there was no need for market imperfections to explain business cycles.

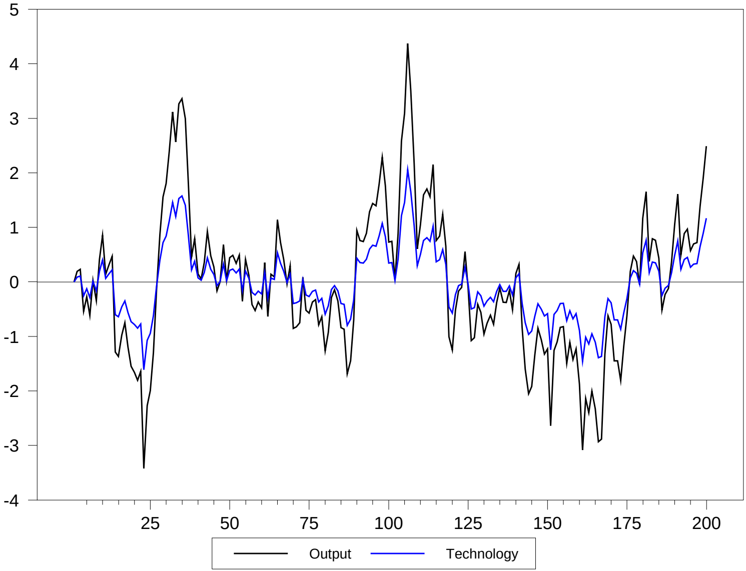
RBC Models Can Generate Cycles with Volatile Investment



The RBC Model's Propagation Mechanisms

- Despite this success, these RBC models have still come in for some criticism.
- One reason is that they have not quite lived up to the hype of their early advocates. Part of that hype stemmed from the idea that RBC models contained important *propagation mechanisms* for turning technology shocks into business cycles.
- The idea was that increases in technology induced extra output through higher capital accumulation and by inducing people to work more.
- In other words, some of the early research suggested that even in a world of iid technology levels, one would expect RBC models to still generate business cycles.
- However, the figure on the next page shows that output fluctuations in this model follow technology fluctuations quite closely: This shows that these additional propagation mechanisms are quite weak.

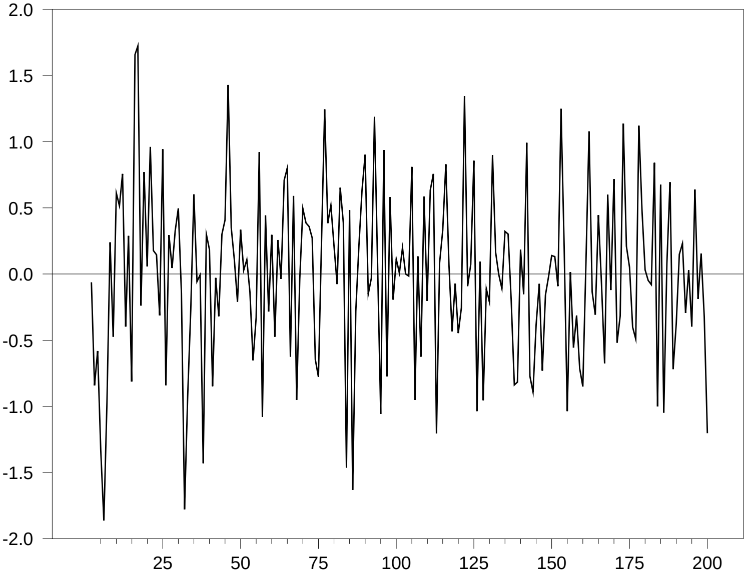
RBC Cycles Rely Heavily on Technology Fluctuations



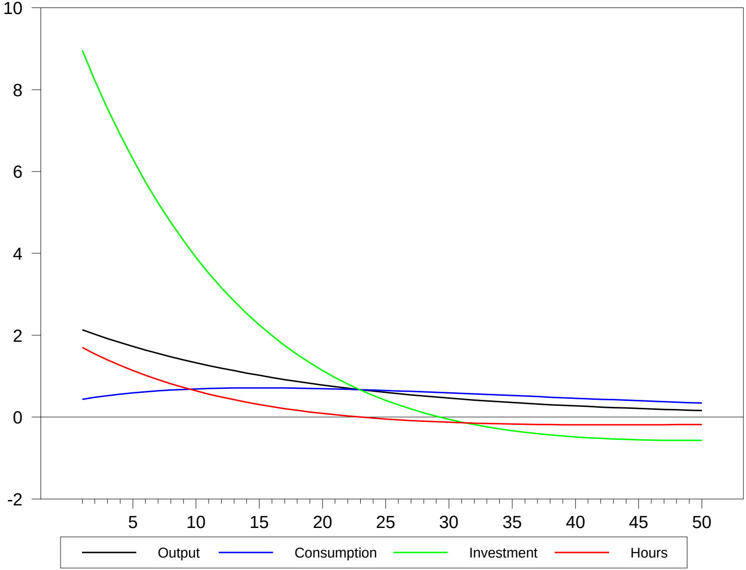
Autocorrelated Growth and Hump-Shaped IRFs

- Cogley and Nason (*AER*, 1995) noted another fact about business cycles that the RBC model does not match: Output growth is positively autocorrelated (not very—autocorrelation coefficient of 0.34—but statistically significant).
- But RBC models do not generate this pattern: See the figure on the next page. They can only do so if one simulates a technology process that has a positively autocorrelated growth rate.
- Cogley and Nason relate this back to the IRFs generated by RBC models. The figure on page 36 shows the responses of output, consumption, investment, and hours to a unit shock to ϵ_t.
- The figure on page 37 highlights that the response of output to the technology shock pretty much matches the response of technology itself.
- Cogley-Nason argue that one needs instead to have "humped-shaped" responses to shocks—a growth rate increase needs to be followed by another growth rate increase—if a model is to match the facts about autocorrelated output growth. The responses to technology shocks do not deliver this. Also, while we don't have other shocks in the model (e.g. government spending shocks), Cogley-Nason show RBC models don't generate hump-shaped responses for these either.

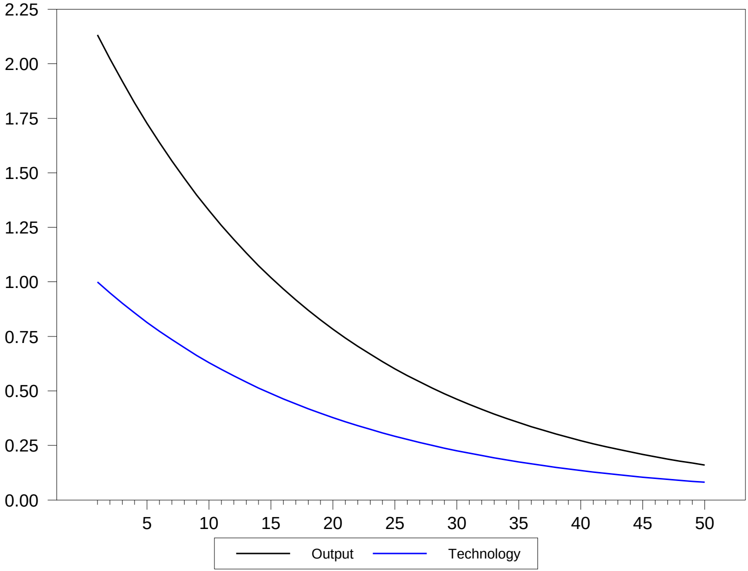
RBCs Do Not Generate Positively Autocorrelated Growth



Impulse Response Functions to Technology Shock



Impulse Response Functions to Technology Shock



Extending the RBC Approach

- In addition to the Cogley-Nason critique, RBC models have also been criticised by Jordi Gali for failing to explain the labour market response to technology shocks.
- Gali has used VARs to show that hours worked tends to decline after a positive technology shock in strong contrast to the model's predictions.
- There are currently a number of branches of research aimed at fixing the deficiencies of the basic RBC approach.
- Some of them involve putting extra bells and whistles on the basic market-clearing RBC approach: Examples include variable utilization, lags in investment projects, habit persistence in consumer utility. Adding these elements tends to strengthen the propagation mechanism element of the model.
- The second approach is to depart more systematically from the basic RBC approach by adding rigidities such as sticky prices and wages. Some papers do this and add the other bells-and-whistles. We will introduce a "full blown" DSGE model later in the course.