

## The Solow Model

We have discussed how economic growth can come from either capital deepening (increased amounts of capital per worker) or from improvements in total factor productivity (sometimes termed technological progress). This suggests that economic growth can come about from saving and investment (so that the economy accumulates more capital) or from improvements in productive efficiency. In these notes, we consider a model that explains the role these two elements play in generating sustained economic growth. The model is also due to Robert Solow, whose work on growth accounting we discussed in the last lecture, and was first presented in his 1956 paper “A Contribution to the Theory of Economic Growth.”

### The Solow Model’s Assumptions

The Solow model assumes that output is produced using a production function in which output depends upon capital and labour inputs as well as a technological efficiency parameter,  $A$ .

$$Y_t = AF(K_t, L_t) \tag{1}$$

It is assumed that adding capital and labour raises output

$$\frac{\partial Y_t}{\partial K_t} > 0 \tag{2}$$

$$\frac{\partial Y_t}{\partial L_t} > 0 \tag{3}$$

However, the model also assumes there are diminishing marginal returns to capital accumulation. In other words, adding extra amounts of capital gives progressively smaller and smaller increases in output. This means the second derivative of output with respect to capital is negative.

$$\frac{\partial^2 Y_t}{\partial K_t^2} < 0 \tag{4}$$

See Figure 1 for an example of how output can depend on capital with diminishing returns. Think about why diminishing marginal returns is probably sensible: If a firm acquires an extra unit of capital, it will probably be able to increase its output. But if the firm keeps piling on extra capital without raising the number of workers available to use this capital, the increases in output will probably taper off. A firm with ten workers would probably like to have at least ten computers. It might even be helpful to have a few more; perhaps a few laptops for work from home or some spare computers in case others break down. But at some point, just adding more computers doesn't help so much.

We will use a very stylized description of the other parts of this economy: This helps us to focus in on the important role played by diminishing marginal returns to capital. We assume a closed economy with no government sector or international trade. This means all output takes the form of either consumption or investment

$$Y_t = C_t + I_t \tag{5}$$

and that savings equals investment

$$S_t = Y_t - C_t = I_t \tag{6}$$

The economy's stock of capital is assumed to change over time according to

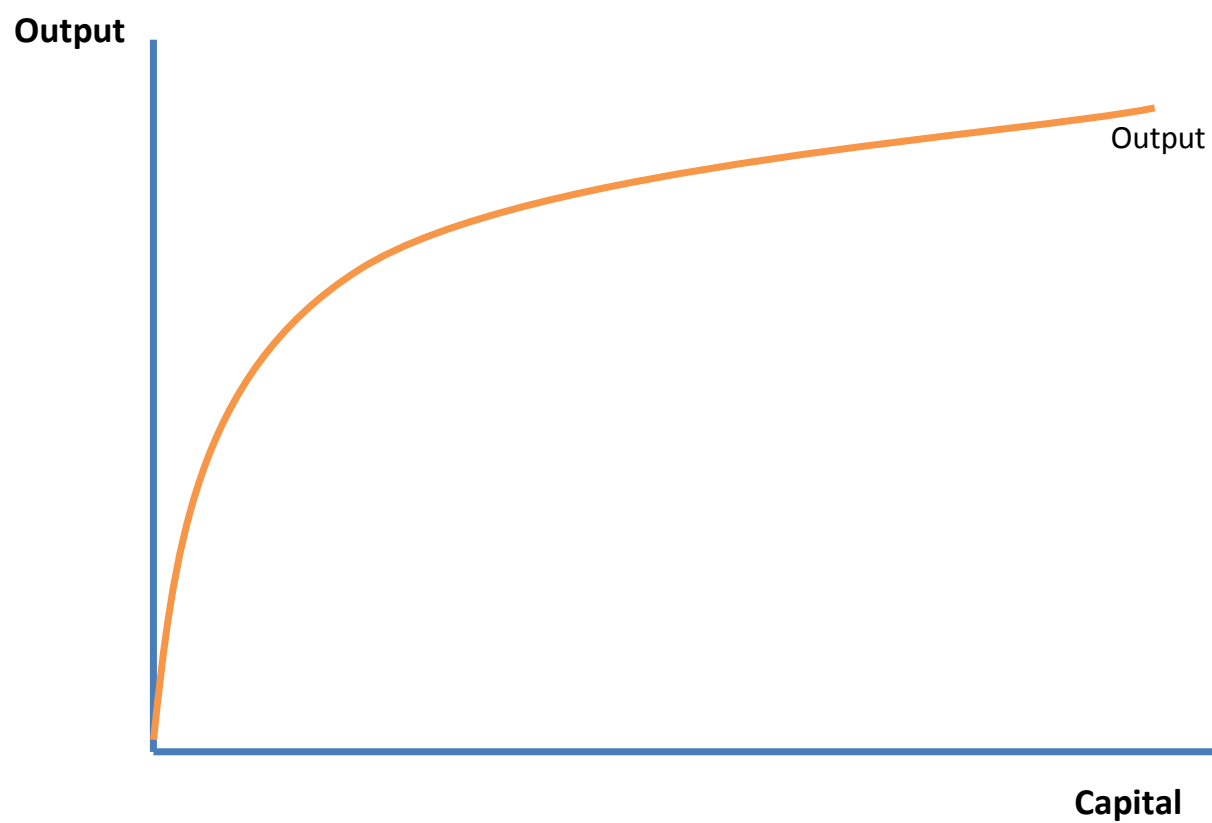
$$\frac{dK_t}{dt} = I_t - \delta K_t \tag{7}$$

In other words, the addition to the capital stock each period depends positively on investment and negatively on depreciation, which is assumed to take place at rate  $\delta$ .

The Solow model does not attempt to model the consumption-savings decision. Instead it assumes that consumers save a constant fraction  $s$  of their income

$$S_t = sY_t \tag{8}$$

**Figure 1: Diminishing Marginal Returns to Capital**



### Capital Dynamics in the Solow Model

Because savings equals investment in the Solow model, equation (8) means that investment is also a constant fraction of output

$$I_t = sY_t \quad (9)$$

which means we can re-state the equation for changes in the stock of capital

$$\frac{dK_t}{dt} = sY_t - \delta K_t \quad (10)$$

Whether the capital stock expands, contracts or stays the same depends on whether investment is greater than, equal to or less than depreciation.

$$\frac{dK_t}{dt} > 0 \quad \text{if} \quad \delta K_t < sY_t \quad (11)$$

$$\frac{dK_t}{dt} = 0 \quad \text{if} \quad \delta K_t = sY_t \quad (12)$$

$$\frac{dK_t}{dt} < 0 \quad \text{if} \quad \delta K_t > sY_t \quad (13)$$

In other words, if the ratio of capital to output is such that

$$\frac{K_t}{Y_t} = \frac{s}{\delta} \quad (14)$$

the the stock of capital will stay constant. If the capital-ouput ratio is lower than this level, then the capital stock will be increasing and if it is higher than this level, it will be decreasing.

Figure 2 provides a graphical illustration of this process. Depreciation is a simple straight-line function of the stock of capital while output is a curved function of capital, featuring diminishing marginal returns. When the level of capital is low  $sY_t$  is greater than  $\delta K$ . As the capital stock increases, the additional investment due to the extra output tails off but the additional depreciation does not, so at some point  $sY_t$  equals  $\delta K$  and the stock of capital stops increasing. Figure 2 labels the particular point at which the capital stock remains unchanged as  $K^*$ . At this point, we have  $\frac{K_t}{Y_t} = \frac{s}{\delta}$ .

In the same way, if we start out with a high stock of capital, then depreciation,  $\delta K$ , will tend to be greater than investment,  $sY_t$ . This means the stock of capital will decline. When it reaches  $K^*$  it will stop declining. This is an example of what economists call *convergent dynamics*. For any fixed set of the model parameters ( $s$  and  $\delta$ ) and other inputs into the production ( $A_t$  and  $L_t$ ) there will be a defined level of capital such that, no matter where the capital stock starts, it will converge over time towards this level.

Figure 3 provides an illustration of how the convergent dynamics determine the level of output in the Solow model. It shows output, investment and depreciation as a function of the capital stock. The gap between the green line (investment) and the orange line (output) shows the level of consumption. The economy converges towards the level of output associated with the capital stock  $K^*$ .

### Convergence Dynamics in Practice

The Solow model predicts that no matter what the original level of capital an economy starts out with, it will tend to revert to the equilibrium levels of output and capital indicated by the economy's underlying features. Does the evidence support this idea?

Unfortunately, history has provided a number of extreme examples of economies having far less capital than is consistent with their fundamental features. Wars have provided the “natural experiments” in which various countries have had huge amounts of their capital destroyed. The evidence has generally supported Solow's prediction that economies that experience negative shocks should tend to recover from these setbacks and return to their pre-shock levels of capital and output. For example, both Germany and Japan grew very strongly after the war, recovering prosperity despite the massive damage done to their stocks

of capital by war bombing.

A more extreme example, perhaps, is study by Edward Miguel and Gerard Roland of the long-run impact of U.S. bombing of Vietnam in the 1960s and 1970s. Miguel and Roland found large variations in the extent of bombing across the various regions of Vietnam. Despite large differences in the extent of damage inflicted on different regions, Miguel and Roland found little evidence for lasting relative damage on the most-bombed regions by 2002. (Note this is not the same as saying there was no damage to the economy as a whole — the study is focusing on whether those areas that lost more capital than average ended up being poorer than average).

### **An Increase in the Savings Rate**

Now consider what happens when the economy has settled down at an equilibrium unchanging level of capital  $K_1$  and then there is an increase in the savings rate from  $s_1$  to  $s_2$ .

Figure 4 shows what happens to the dynamics of the capital stock. The line for investment shifts upwards: For each level of capital, the level of output associated with it translates into more investment. So the investment curve shifts up from the green line to the red line. Starting at the initial level of capital,  $K_1$ , investment now exceeds depreciation. This means the capital stock starts to increase. This process continues until capital reaches its new equilibrium level of  $K_2$  (where the red line for investment intersects with the black line for depreciation.) Figure 5 illustrates how output increases after this increase in the savings rate.

Figure 2: Capital Dynamics in The Solow Model

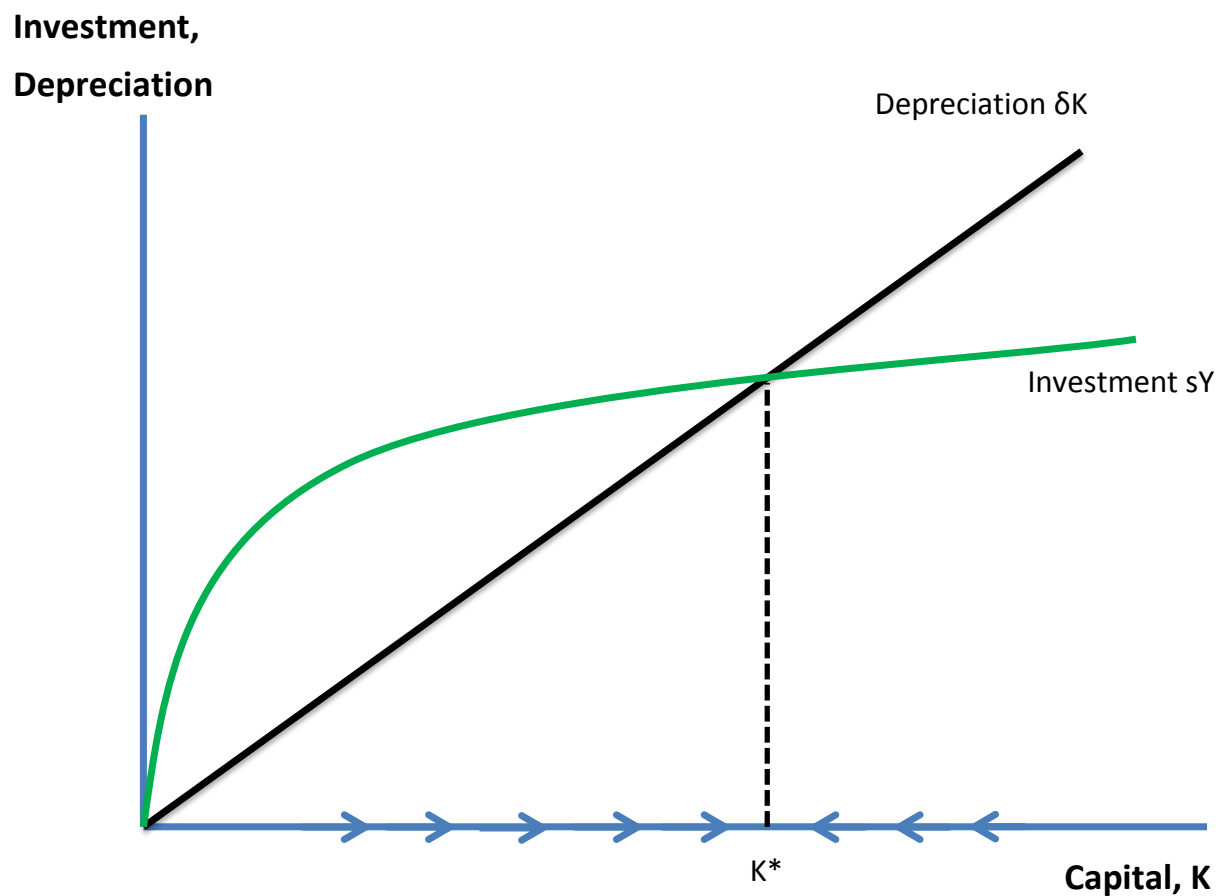


Figure 3: Capital and Output in the Solow Model

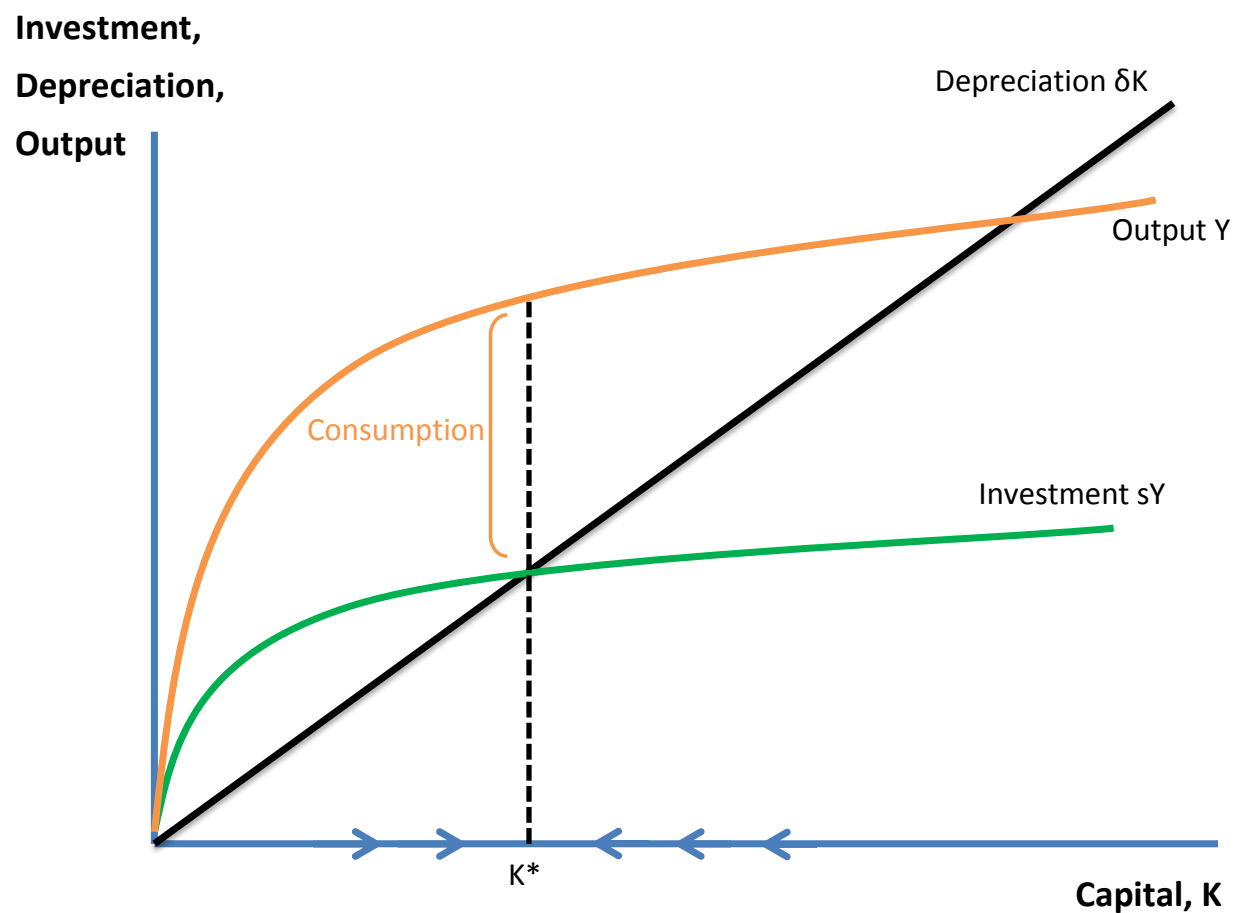




Figure 4: An Increase in the Saving Rate

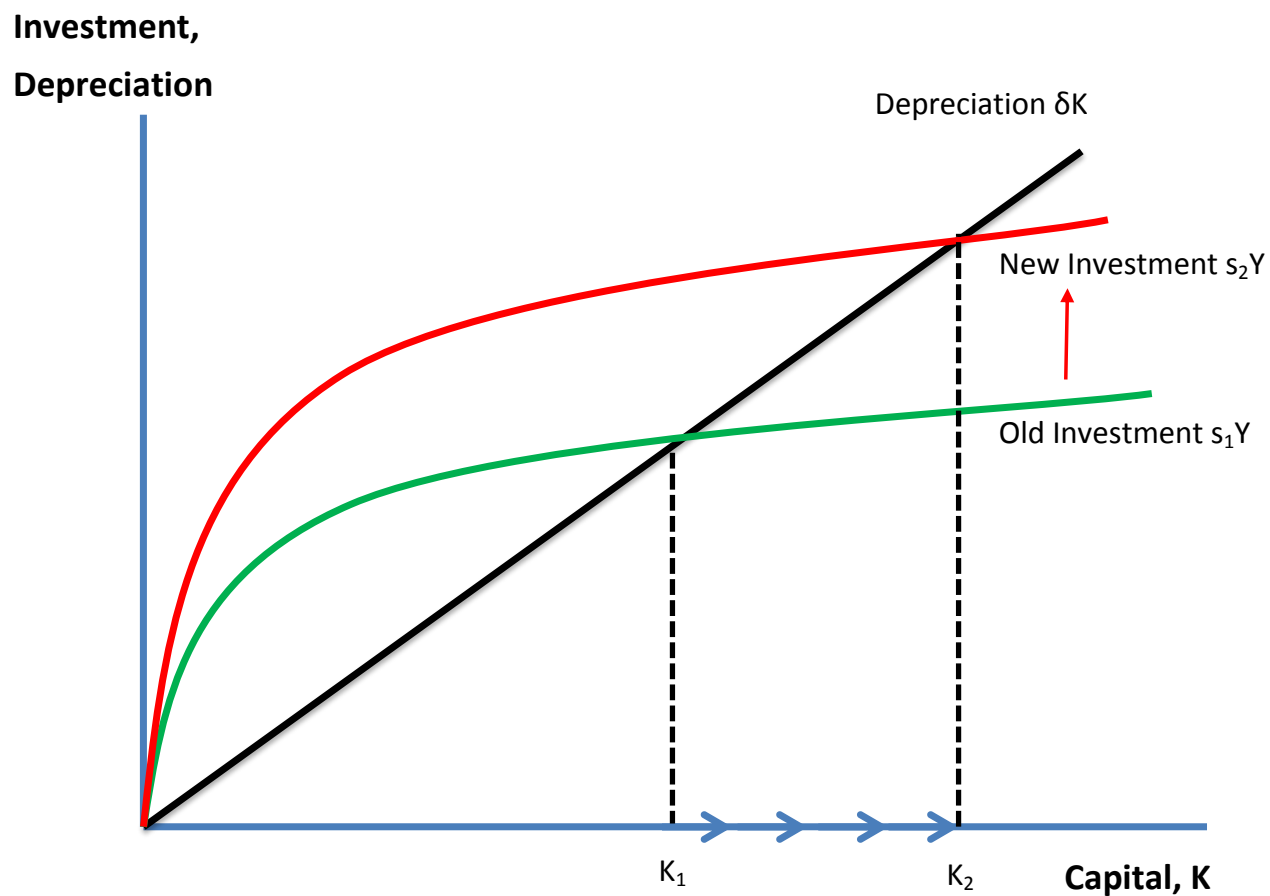
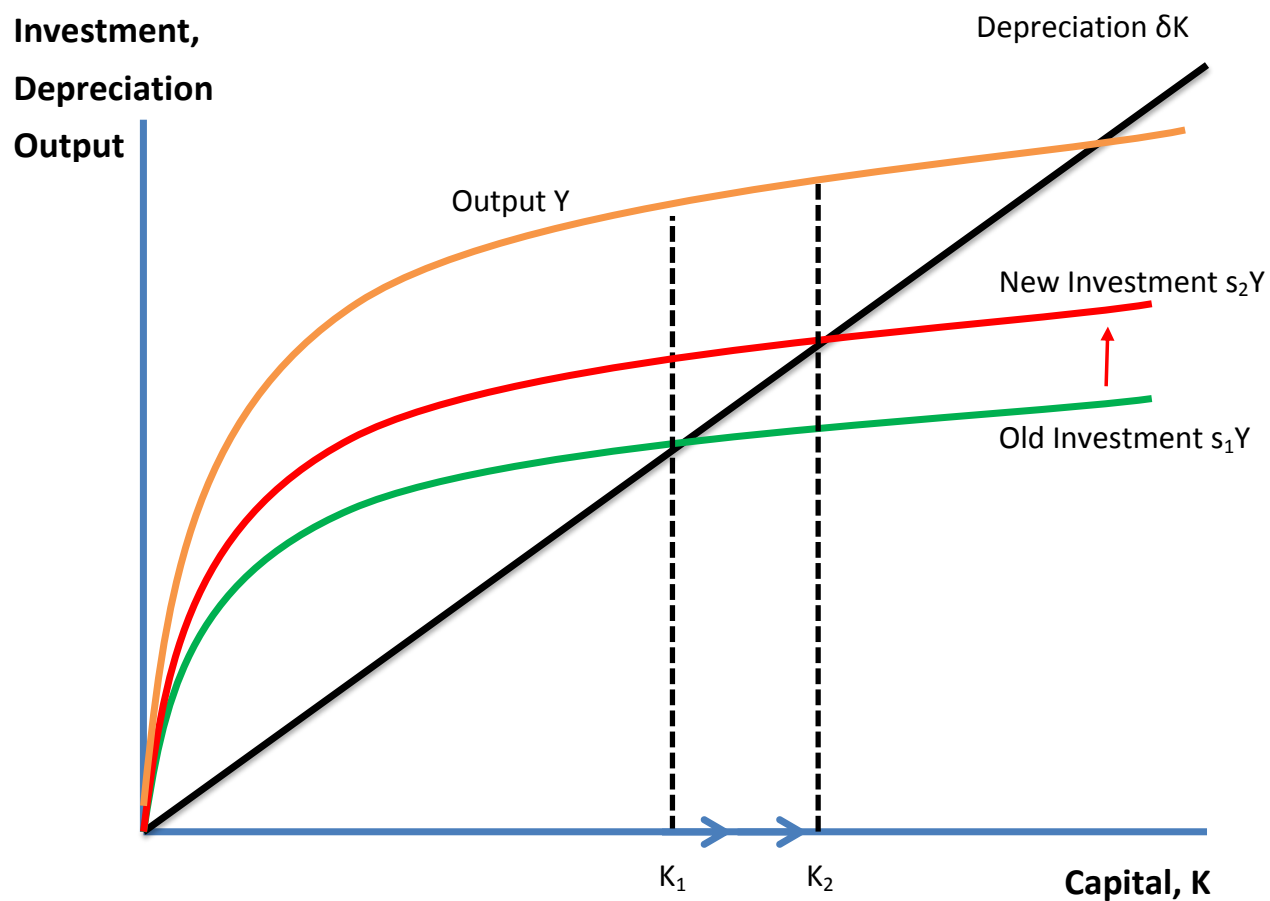


Figure 5: Effect on Output of Increased Saving



### **An Increase in the Depreciation Rate**

Now consider what happens when the economy has settled down at an equilibrium unchanging level of capital  $K_1$  and then there is an increase in the depreciation rate from  $\delta_1$  to  $\delta_2$ .

Figure 6 shows what happens in this case. The depreciation schedule shifts up from the black line associated with the original depreciation rate,  $\delta_1$ , to the new red line associated with the new depreciation rate,  $\delta_2$ . Starting at the initial level of capital,  $K_1$ , depreciation now exceeds investment. This means the capital stock starts to decline. This process continues until capital falls to its new equilibrium level of  $K_2$  (where the red line for depreciation intersects with the green line for investment.) So the increase in the depreciation rate leads to a decline in the capital stock and in the level of output.

### **An Increase in Technological Efficiency**

Now consider what happens when technological efficiency  $A_t$  increases. Because investment is given by

$$I_t = sY_t = sAF(K_t, L_t) \tag{15}$$

a one-off increase in  $A$  thus has the same effect as a one-off increase in  $s$ . Capital and output gradually rise to a new higher level. Figure 7 shows the increase in capital due to an increase in technological efficiency.

**Figure 6: An Increase in Depreciation**

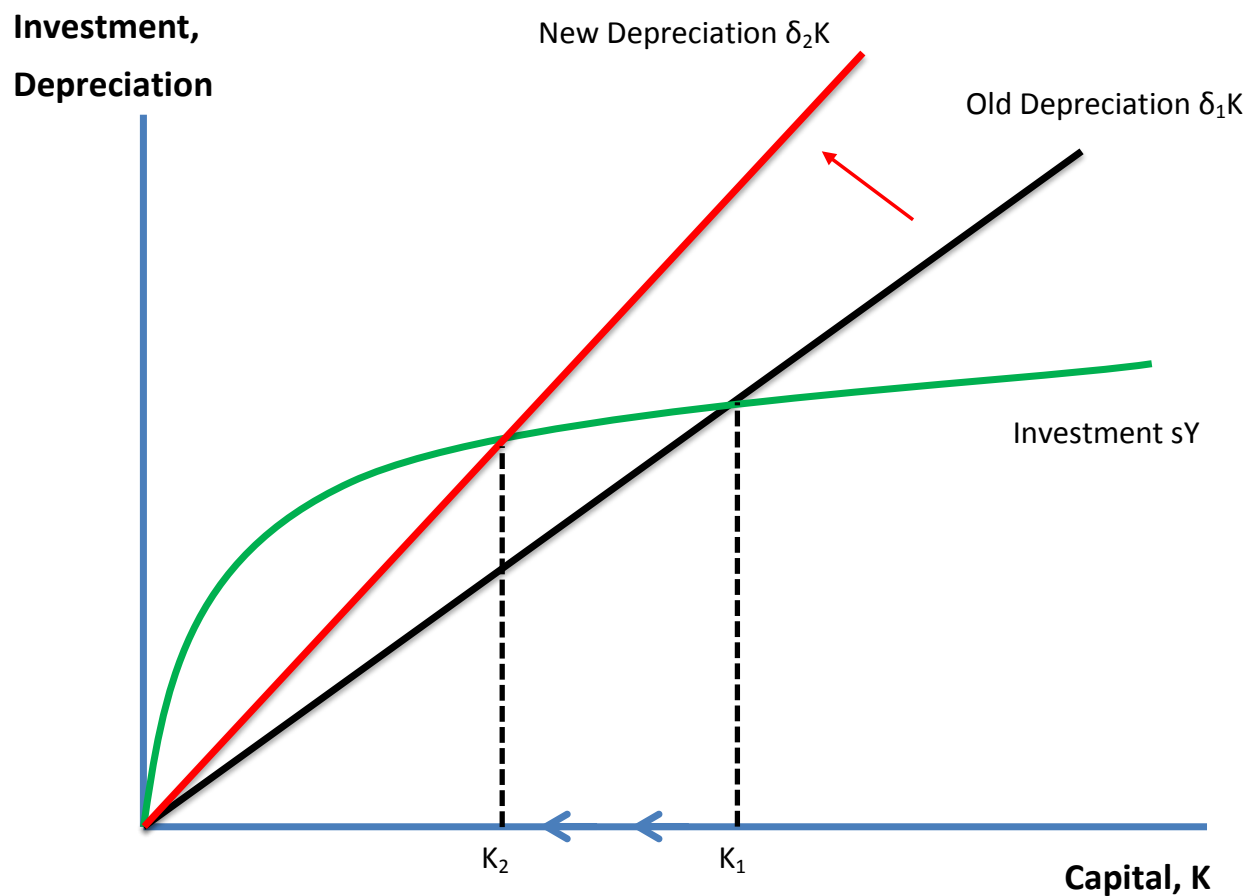
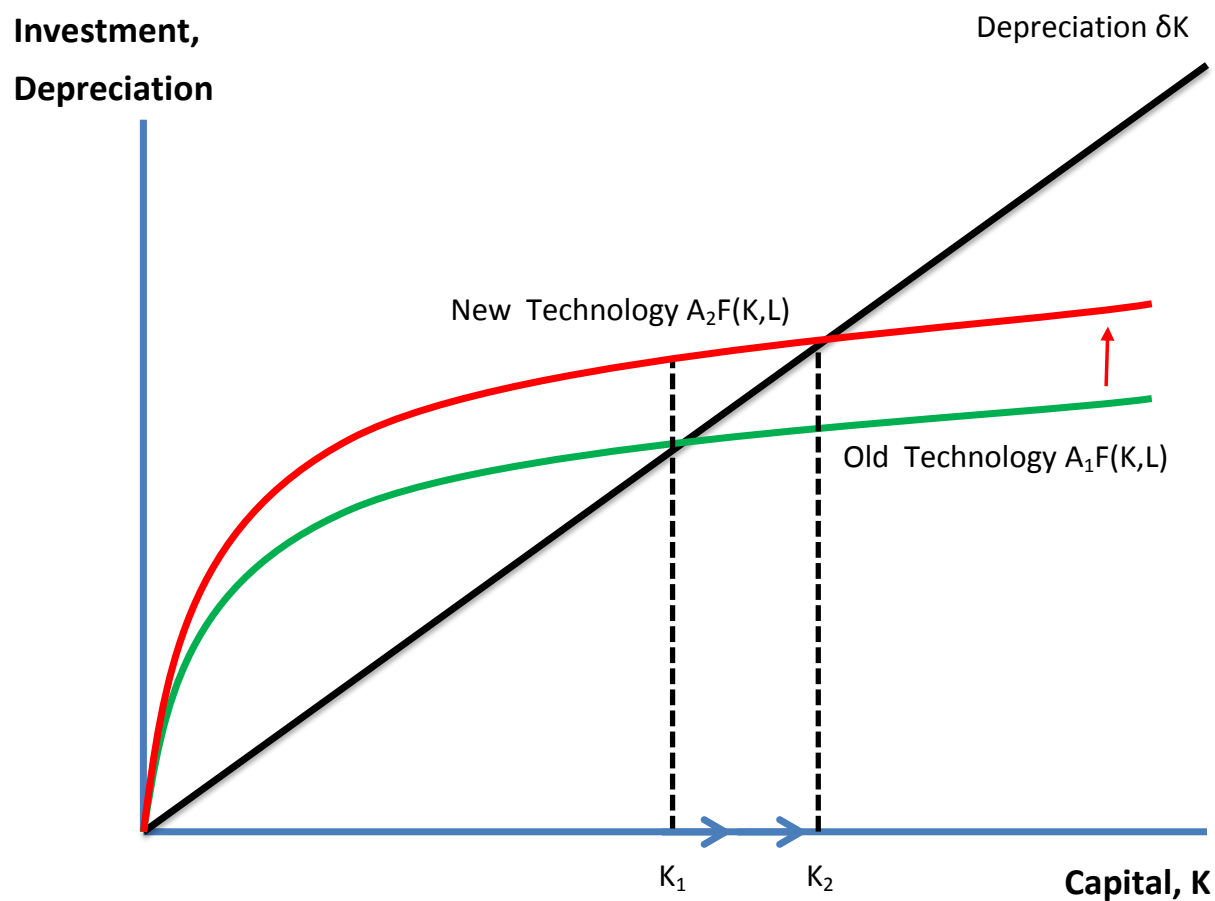


Figure 7: An Increase in Technological Efficiency



## Solow and the Sources of Growth

In the last lecture, we described how capital deepening and technological progress were the two sources of growth in output per worker. Specifically, we derived an equation in which output growth was a function of growth in the capital stock, growth in the number of workers and growth in technological efficiency.

Our previous discussion had pointed out that a one-off increase in technological efficiency,  $A_t$ , had the same effects as a one-off increase in the savings rate,  $s$ . However, there are important differences between these two types of improvements. The Solow model predicts that economies can only achieve a temporary boost to economic growth due to a once-off increase in the savings rate. If they want to sustain economic growth through this approach, then they will need to keep raising the savings rate. However, there are likely to be limits in any economy to the fraction of output that can be allocated towards saving and investment, particularly if it is a capitalist economy in which savings decisions are made by private citizens.

Unlike the savings rate, which will tend to have an upward limit, there is no particular reason to believe that technological efficiency  $A_t$  has to have an upper limit. Indeed, growth accounting studies tend to show steady improvements over time in  $A_t$  in most countries. Going back to Young's paper on Hong Kong and Singapore discussed in the last lecture, you can see now why it matters whether an economy has grown due to capital deepening or TFP growth. The Solow model predicts that a policy of encouraging growth through more capital accumulation will tend to tail off over time producing a once-off increase in output per worker. In contrast, a policy that promotes the growth rate of TFP can lead to a sustained higher growth rate of output per worker.

### The Capital-Output Ratio with Steady Growth

Up to now, we have only considered once-off changes in output. Here, however, we consider how the capital stock behaves when the economy grows at steady constant rate  $G^Y$ . Specifically, we can show in this case that the ratio of capital to output will tend to converge to a specific value. Recall from the last lecture that if we have something of the form

$$Z_t = U_t^\alpha W_t^\beta \quad (16)$$

then we have the following relationship between the various growth rates

$$G_t^Z = \alpha G_t^U + \beta G_t^W \quad (17)$$

The capital output ratio  $\frac{K_t}{Y_t}$  can be written as  $K_t Y_t^{-1}$ . So the growth rate of the capital-output ratio can be written as

$$G_t^{\frac{K}{Y}} = G_t^K - G_t^Y \quad (18)$$

Adjusting equation 10, the growth rate of the capital stock can be written as

$$G_t^K = \frac{1}{K_t} \frac{dK_t}{dt} = s \frac{Y_t}{K_t} - \delta \quad (19)$$

so the growth rate of the capital-output ratio is

$$G_t^{\frac{K}{Y}} = s \frac{Y_t}{K_t} - \delta - G^Y \quad (20)$$

This gives a slightly different form of convergence dynamics from those we saw earlier. This equation shows that the growth rate of the capital-output ratio depends negatively on the level of this ratio. This means the capital-output ratio displays convergent dynamics. When it is above a specific equilibrium value it tends to fall and when it is below this equilibrium value it tends to increase. Thus, the ratio is constantly moving towards this equilibrium value.

We can express this formally as follows:

$$G_t^{\frac{K}{Y}} > 0 \quad \text{if} \quad \frac{K_t}{Y_t} < \frac{s}{\delta + G^Y} \quad (21)$$

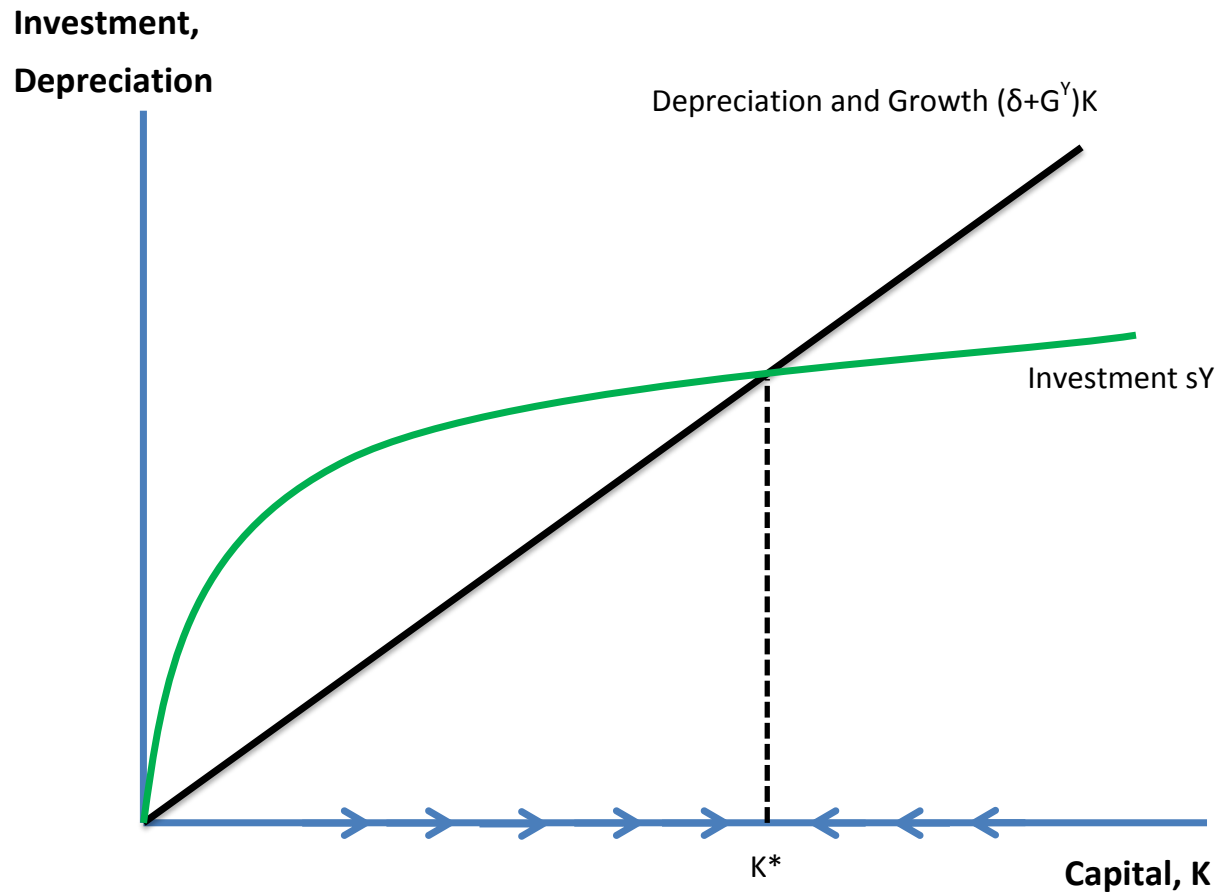
$$G_t^{\frac{K}{Y}} = 0 \quad \text{if} \quad \frac{K_t}{Y_t} = \frac{s}{\delta + G^Y} \quad (22)$$

$$G_t^{\frac{K}{Y}} < 0 \quad \text{if} \quad \frac{K_t}{Y_t} > \frac{s}{\delta + G^Y} \quad (23)$$

We can illustrate these dynamics using a slightly altered version of our earlier graph. Figure 8 amends the depreciation line to the amount of capital necessary not just to replace depreciation but also to have a percentage increase in the capital stock that matches the increase in output. The diagram shows that the economy will tend to move towards a capital stock such that  $sY_t = (\delta + G^Y) K_t$  meaning the capital-output ratio is  $\frac{K_t}{Y_t} = \frac{s}{\delta + G^Y}$ .



**Figure 8: The Equilibrium Capital Stock in a Growing Economy**



**Briefly, Back to Piketty**

We previously discussed one of Thomas Piketty's explanations for why capital may tend to grow faster than income (the  $r > g$  argument). Piketty has a different argument for why capital may grow faster than income that relates to the result we have just derived.

In his book, Piketty describes a different assumption about savings in the economy from the one we have just derived. Specifically, he works with a net savings rate,  $\tilde{s}$ , which is defined as follows

$$I_t - \delta K_t = \tilde{s} Y_t \quad (24)$$

In other words, defined like this,  $\tilde{s}$  is a savings rate that subtracts off the share of GDP taken up by capital depreciation. In the same way, net national product is defined as GDP minus depreciation. Given this definition, we can write the change in the capital stock as

$$\Delta K_t = \tilde{s} Y_t \quad (25)$$

Repeating the calculations from above with this model, the growth rate of capital

$$G_t^K = \frac{1}{K_t} \frac{dK_t}{dt} = \frac{I_t - \delta K_t}{K_t} \quad (26)$$

becomes

$$G_t^K = \frac{1}{K_t} \frac{dK_t}{dt} = \tilde{s} \frac{Y_t}{K_t} \quad (27)$$

So the growth rate of the capital-output ratio is

$$G_t^{\frac{K}{Y}} = \tilde{s} \frac{Y_t}{K_t} - G_t^Y \quad (28)$$

This means we can write convergence dynamics in terms of this net savings rate.

$$G_t^{\frac{K}{Y}} > 0 \quad \text{if} \quad \frac{K_t}{Y_t} < \frac{\tilde{s}}{G^Y} \quad (29)$$

$$G_t^{\frac{K}{Y}} = 0 \quad \text{if} \quad \frac{K_t}{Y_t} = \frac{\tilde{s}}{G^Y} \quad (30)$$

$$G_t^{\frac{K}{Y}} < 0 \quad \text{if} \quad \frac{K_t}{Y_t} > \frac{\tilde{s}}{G^Y} \quad (31)$$

So the capital output ratio converges to  $\frac{K_t}{Y_t} = \frac{\tilde{s}}{G^Y}$ . Again showing his gift for grand terminology, Piketty calls this result *the second fundamental law of capitalism*. His research has argued that growth appears to be slowing around the world and thus, with  $G^Y$  in the denominator heading towards zero, the capital-output ratio is likely to be keep rising.

The idea that slow growth will raise the capital-output ratio also holds for the standard Solow model formulation in which the capital-output ratio converges to  $\frac{s}{\delta + G^Y}$ . The big difference here, however, is that Piketty's formulation suggests that when  $G^Y$  tends towards zero that we could see the capital-output ratio head towards infinity because his steady-state ratio does not have the  $\delta$  in the denominator. However, this is somewhat misleading. In the standard formulation of the model, you can show that the net savings rate along a steady growth path will be

$$\tilde{s} = \frac{I_t}{Y_t} - \delta \frac{K_t}{Y_t} \quad (32)$$

$$= s - \frac{s\delta}{G_Y + \delta} \quad (33)$$

$$= \frac{sG_Y}{G_Y + \delta} \quad (34)$$

So when output growth goes to zero, the net savings rate also goes to zero. This means we shouldn't just look at Piketty's formula of  $\frac{\tilde{s}}{G^Y}$  for the steady-state capital-output ratio and imagine the denominator ( $G_Y$ ) heading to zero while the numerator  $\tilde{s}$  is fixed. From this discussion, you can take that slower output growth is likely to raise the ratio of capital to income, but it is not likely to head towards infinity!

### A Formula for Steady Growth

As a last calculation, we can derive a formula for the growth rate of an economy that is expanding at a constant rate. Let output be determined by a Cobb-Douglas production function

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha} \quad (35)$$

This means output growth is determined by

$$G_t^Y = G_t^A + \alpha G_t^K + (1 - \alpha) G_t^L \quad (36)$$

Now consider the case in which the growth rate of labour input is fixed at  $n$

$$G_t^L = n \quad (37)$$

and the growth rate of total factor productivity is fixed at  $g$ .

$$G_t^A = g \quad (38)$$

The formula for output growth becomes

$$G_t^Y = g + \alpha G_t^K + (1 - \alpha) n \quad (39)$$

This means all variations in the growth rate of output are due to variations in the growth rate for capital. If output is growing at a constant rate, then capital must also be growing at a constant rate. And we know that the capital-output ratio tends to move towards a specific equilibrium value. So along a steady growth path, the growth rate of output equals the growth rate of capital. Thus, the previous equation can be re-written

$$G_t^Y = g + \alpha G_t^Y + (1 - \alpha) n \quad (40)$$

which can be simplified to

$$G_t^Y = \frac{g}{1 - \alpha} + n \quad (41)$$

The growth rate of output per worker is

$$G_t^Y - n = \frac{g}{1 - \alpha} \quad (42)$$

So the economy tends to converge towards a steady growth path and the growth rate of output per worker along this path is  $\frac{g}{1-\alpha}$ . Without growth in technological efficiency, there can be no steady growth in output per worker.

### **Why Growth Accounting Can Be Misleading**

Of the cases just considered in which output and capital both increase—an increase in the savings rate and an increase in the level of TFP—the evidence points to increases in TFP being more important as a generator of long-term growth. Rates of savings and investment tend for most countries tend to stay within certain ranges while large increases in TFP over time have been recorded for many countries. It's worth noting then that growth accounting studies can perhaps be a bit misleading when considering the ultimate sources of growth.

Consider a country that has a constant share of GDP allocated to investment but is experiencing steady growth in TFP. The Solow model predicts that this economy should experience steady increases in output per worker and increases in the capital stock. A growth accounting exercise may conclude that a certain percentage of growth stems from capital accumulation but ultimately, in this case, all growth (including the growth in the capital stock) actually stems from growth in TFP. The moral here is that pure accounting exercises may miss the ultimate cause of growth.

### **Krugman on “The Myth of Asia’s Miracle”**

I encourage you to read Paul Krugman’s 1994 article “The Myth of Asia’s Miracle.” It discusses a number of examples of cases where economies where growth was based on largely on capital accumulation. In addition to the various Asian countries covered in Alwyn Young’s research, Krugman (correctly) predicted a slowdown in growth in Japan, even though at the time many US commentators were focused on the idea that Japan was going to overtake US levels of GDP per capita.

Perhaps most interesting is his discussion of growth in the Soviet Union. Krugman notes that the Soviet economy grew strongly after World War 2 and many in the West believed they would become more prosperous than capitalist economies. The Soviet Union’s achievement in placing the first man in space provoked Kennedy’s acceleration in the space programme, mainly to show the U.S. was not falling behind communist systems. However, some economists that had examined the Soviet economy were less impressed. Here’s an extended quote from Krugman’s article:

When economists began to study the growth of the Soviet economy, they did so using the tools of growth accounting. Of course, Soviet data posed some problems. Not only was it hard to piece together usable estimates of output and input (Raymond Powell, a Yale professor, wrote that the job “in many ways resembled an archaeological dig”), but there were philosophical difficulties as well. In a socialist economy one could hardly measure capital input using market returns, so researchers were forced to impute returns based on those in market economies at similar levels of development. Still, when the efforts began, researchers were pretty sure about what: they would find. Just as capitalist growth had been based on

growth in both inputs and efficiency, with efficiency the main source of rising per capita income, they expected to find that rapid Soviet growth reflected both rapid input growth and rapid growth in efficiency.

But what they actually found was that Soviet growth was based on rapid-growth in inputs—end of story. The rate of efficiency growth was not only unspectacular, it was well below the rates achieved in Western economies. Indeed, by some estimates, it was virtually nonexistent.

The immense Soviet efforts to mobilize economic resources were hardly news. Stalinist planners had moved millions of workers from farms to cities, pushed millions of women into the labor force and millions of men into longer hours, pursued massive programs of education, and above all plowed an ever-growing proportion of the country's industrial output back into the construction of new factories.

Still, the big surprise was that once one had taken the effects of these more or less measurable inputs into account, there was nothing left to explain. The most shocking thing about Soviet growth was its comprehensibility.

This comprehensibility implied two crucial conclusions. First, claims about the superiority of planned over market economies turned out to be based on a misapprehension. If the Soviet economy had a special strength, it was its ability to mobilize resources, not its ability to use them efficiently. It was obvious to everyone that the Soviet Union in 1960 was much less efficient than the United States. The surprise was that it showed no signs of closing the gap.

Second, because input-driven growth is an inherently limited process, Soviet growth was virtually certain to slow down. Long before the slowing of Soviet growth be-

came obvious, it was predicted on the basis of growth accounting.

The Soviet leadership did a good job for a long time of hiding from the world that their economy had stopped growing but ultimately the economic failures of the centrally planning model (combined with its many political and ethnic tensions) ended in a dramatic implosion of the communist system in Russia and the rest of Eastern Europe.

### **The Future of Growth in Europe and Structural Reforms**

In my paper with Kieran McQuinn, “Europe’s Long-Term Growth Prospects: With and Without Structural Reforms” we use a Solow-style framework to model the prospects for growth in the euro area 12 group of economies (the first twelve members). We describe a scenario in which the euro area economy has a cyclical recovery up to 2020 after which the unemployment rate and investment share of GDP remain at pre-crisis levels. TFP growth in the euro area is projected to be only 0.2 percent per year — the average rate since 2000. With a Cobb-Douglas production function and  $\alpha = \frac{1}{3}$ , that implies a steady-state growth rate for output per unit of labour input of only 0.3 percent per year.

Figure 8 shows how it takes a very long time before the growth rate of the capital stock falls to its steady-state level, so the capital-output ratio is projected to keep increasing for decades and the output per unit of labour input continues to grow at a rate higher than its steady-state level of 0.3 percent. Figure 9 shows how low the growth rate of the euro area economy would be under this scenario: The economy would grow slowly due to poor productivity growth and demographic factors limiting growth in labour input.

Many in Europe call for “structural reforms” to boost growth rates. Some of these reforms are measures to increase the number of people at work such as labour market reforms to



reduce unemployment rates and changes to pension systems to reduce early retirement rates. We show in our paper that while these policies could have a temporary positive impact on growth, they are likely to be relatively small and they depress labour productivity growth: See Figure 10 for our estimate of the impact these policies would have on euro area growth rates. To really boost growth in Europe, structural reforms need to increase the growth rate of TFP.

Figure 8: McQuinn-Whelan Projections for Capital Growth

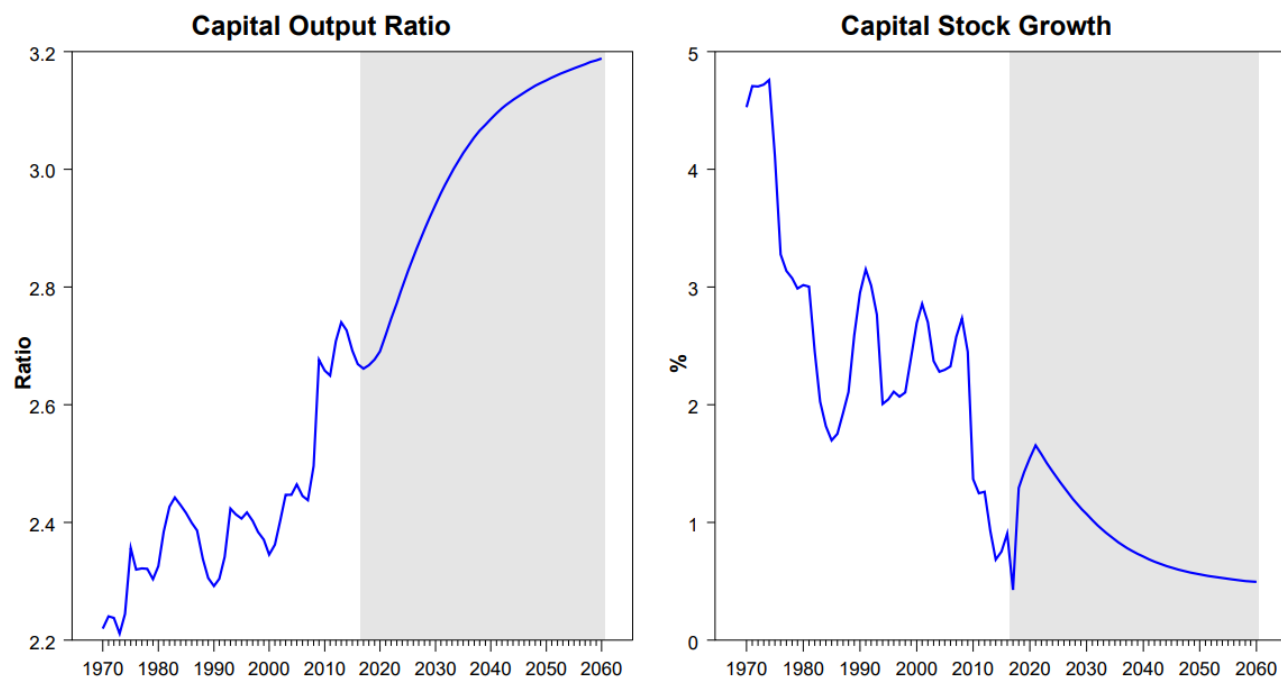
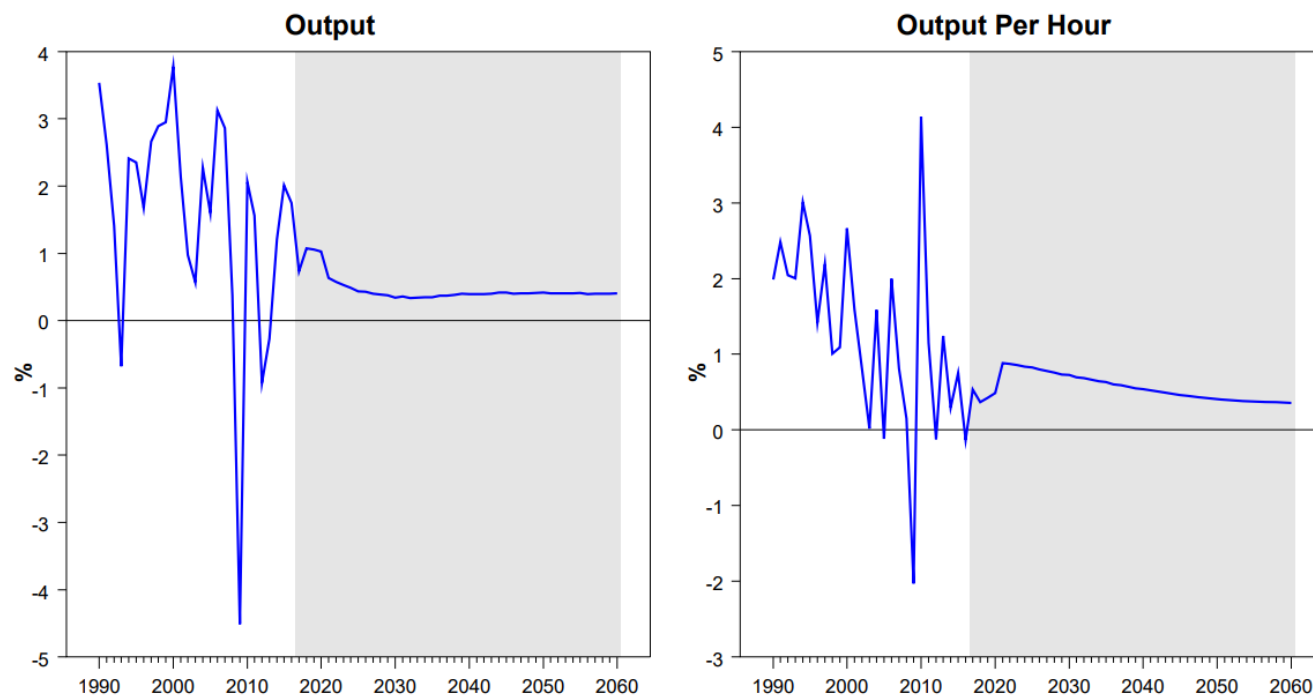
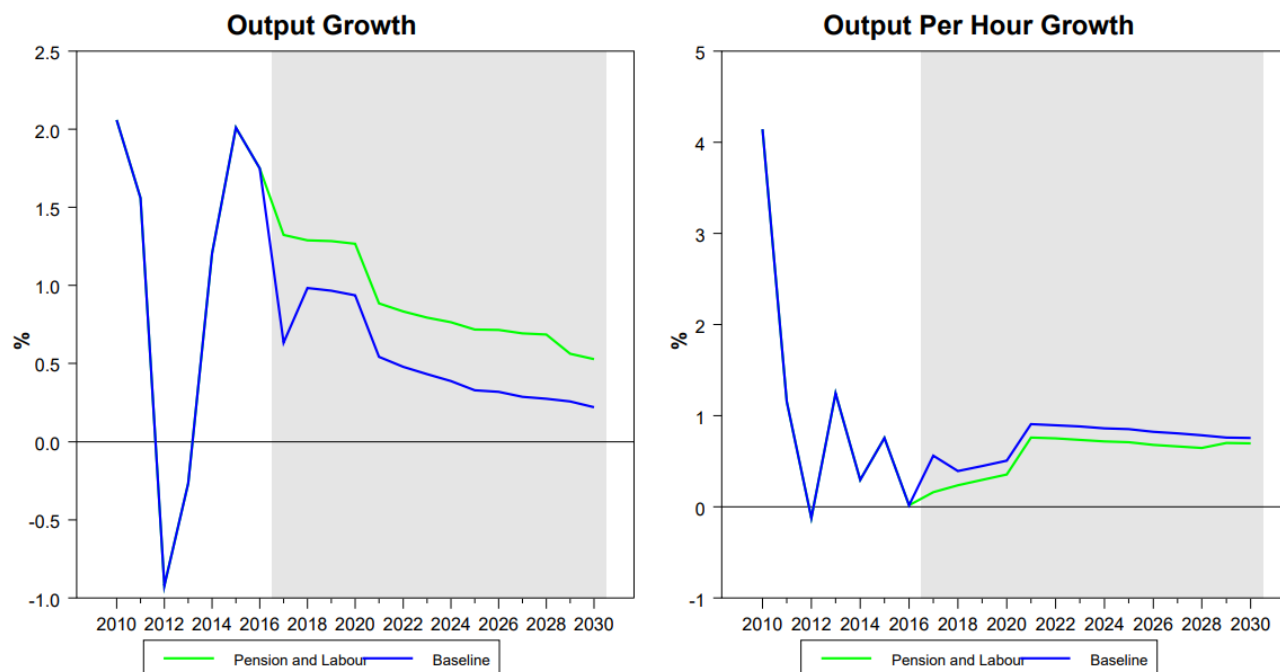


Figure 9: McQuinn-Whelan Projections for Output Growth



**Figure 10: McQuinn-Whelan Estimates of Impact of Reforms**



### **Things to Understand from these Notes**

Here's a brief summary of the things that you need to understand from these notes.

1. The assumptions of the Solow model.
2. The rationale for diminishing marginal returns to capital accumulation.
3. The Solow model's predictions about convergent dynamics.
4. Historical examples of convergent dynamics.
5. Effects of changes in savings rate, depreciation rate and technology in the Solow model.
6. Why technological progress is the source of most growth.
7. Convergence dynamics for the capital-output ratio in a growing economy.
8. Piketty's formula for the steady capital-output ratio.
9. The formula for steady growth rate with a Cobb-Douglas production function.
10. Why growth accounting calculations can underestimate the role of technological progress.
11. Krugman on the Soviet Union.
12. McQuinn and Whelan on the euro area.