# PhD Macroeconomics 1
# 3. Dynamic Optimisation in Continuous Time

Karl Whelan

School of Economics, UCD

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# Part I

## Optimal Control and the Maximum Principle

# Optimal Control Problems

- Optimal control theory describes how to solve problems of the form

$$\underset{\{u_t, x_t\}}{Max} \int_{t_0}^{t_1} F(x_t, u_t, t)\, dt$$

subject to

$$\dot{x}_t = A(x_t, u_t, t)$$

- We will term $x$ the **state variable** and $u$ the **control variable** but in this case, time will be continuous.
- We will go through a relatively informal discussion of the conditions for optimality in these problems. Let's start by treating it as a Lagrangian problem where the law of motion of the state variable gives us a constraint at each point in time. So we set up the Lagrangian as

$$L = \int_{t_0}^{t_1} \left[F(x_t, u_t, t) + \lambda_t \left(A(x_t, u_t, t) - \dot{x}_t\right)\right] dt$$

While clearly a Lagrange multiplier, in the optimal control literature $\lambda_t$ is often referred to as the **costate variable**.
- To maximise this we need to know a few tricks relating to integrals.

# Two Integral Tricks

1. **Leibniz Rule**: If we let

$$I(x) = \int_{t_0}^{t_1} F(x,t)\,dt$$

then

$$\frac{dI}{dx} = \int_{t_0}^{t_1} \frac{\partial F}{\partial x} dt$$

2. **Integration by Parts**: This is the product rule of differentiation backwards.

$$\begin{aligned}
\frac{d}{dx}\left(f(x)\,g(x)\right) &= f'(x)\,g(x) + f(x)\,g'(x) \\
\int_{t_0}^{t_1} \frac{d}{dx}\left(f(x)\,g(x)\right) dx &= \int_{t_0}^{t_1} f'(x)\,g(x)\,dx + \int_{t_0}^{t_1} f(x)\,g'(x)\,dx \\
\implies \int_{t_0}^{t_1} f'(x)\,g(x)\,dx &= f(t_1)\,g(t_1) - f(t_0)\,g(t_0) + \int_{t_0}^{t_1} f(x)\,g'(x)\,dx
\end{aligned}$$

# Optimal Control Problems

- Our Lagrangian is

$$L = \int_{t_0}^{t_1} \left[ F\left(x_t, u_t, t\right) + \lambda_t \left( A\left(x_t, u_t, t\right) - \dot{x}_t \right) \right] dt$$

- Let's use integration by parts to replace the term involving $\dot{x}_t$.

$$- \int_{t_0}^{t_1} \lambda_t \dot{x}_t dt = \lambda_{t_0} x_{t_0} - \lambda_{t_1} x_{t_1} + \int_{t_0}^{t_1} \dot{\lambda}_t x_t dt$$

- So the Lagrangian becomes

$$L = \int_{t_0}^{t_1} \left[ F\left(x_t, u_t, t\right) + \lambda_t A\left(x_t, u_t, t\right) + \dot{\lambda}_t x_t \right] dt + \lambda_{t_0} x_{t_0} - \lambda_{t_1} x_{t_1}$$

- Now we look to maximise this with respect to $x_t$ and $u_t$ point by point. In other words, we are looking to maximise the integral by trying the maximise the height of the function at each point.

# Optimality Condition 1: The Control Variable

- Taking derivatives of

$$L = \int_{t_0}^{t_1} \left[ F\left(x_t, u_t, t\right) + \lambda_t A\left(x_t, u_t, t\right) + \dot{\lambda}_t x_t \right] dt + \lambda_{t_0} x_{t_0} - \lambda_{t_1} x_{t_1}$$

  with respect to $u_t$, the Leibniz rule implies that we maximise the integral by separately maximising the function being integrated at each point in time.

- This means that for all $t \in (t_0, t_1)$, we have

$$\frac{\partial F}{\partial u_t} + \lambda_t \frac{\partial A}{\partial u_t} = 0$$

- We can still interpret $\lambda_t$ as a shadow value, in this case it is the value of the state variable. So this equation states that the direct payoff of an additional unit of the control variable $u_t$ must be exactly offset by the effect this additional unit has in adding or subtracting to the state variable, $x_t$.

# Optimality Condition 2: The State Variable

- Taking derivatives of

$$L = \int_{t_0}^{t_1} \left[ F(x_t, u_t, t) + \lambda_t A(x_t, u_t, t) + \dot{\lambda}_t x_t \right] dt + \lambda_{t_0} x_{t_0} - \lambda_{t_1} x_{t_1}$$

with respect to $x_t$ where $t \in (t_0, t_1)$, we get

$$\frac{\partial F}{\partial x_t} + \lambda_t \frac{\partial A}{\partial x_t} + \dot{\lambda}_t = 0$$

- The first term defines the instantaneous payoff to having more of $x_t$; the second term describes the effect of an additional unit of $x_t$ on the change in $x_t$; the final term accounts for changes in the value of the state variable.

- We also have

$$\lambda_{t_1} = 0$$

At time $t_1$, there is no value to accumulating more of the state variable because payoff after this period are not being counted. This is known as the **transversality condition**.

# The Hamiltonian and the Maximum Principle

- If we define the Hamiltonian of this problem to be

$$H(x_t, u_t, t) = F(x_t, u_t, t) + \lambda_t A(x_t, u_t, t)$$

- Then the following set of conditions will make the path $(x_t^*, u_t^*, \lambda_t^*)$ represent an optimal solution:

$$\begin{aligned}
\frac{\partial H(x_t^*, u_t^*, \lambda_t^*)}{\partial u} &= 0 \\
\frac{\partial H(x_t^*, u_t^*, \lambda_t^*)}{\partial x} &= -\dot{\lambda} \\
\frac{\partial H(x_t^*, u_t^*, \lambda_t^*)}{\partial \lambda} &= \dot{x} \\
\lambda_{t_1} &= 0
\end{aligned}$$

- Together, these four conditions are known as **Pontryagin's maximum principle**.

# Exponential Discounting in Continuous Time

- Most intertemporal problems in economics involve discounting future payoffs. In discrete time is common to see this represented by a discount rate of the form

$$\beta = \frac{1}{1+r}$$

- Suppose interest is accumulated $n$ times during period, with $r/n$ earned each sub-period and the balance compounded. Then, the discount rate would be

$$\beta = \left(\frac{1}{1+\frac{r}{n}}\right)^n$$

- Continuous time can be approximated as the limit case of the number of sub-periods going towards infinity. It turns out however that

$$\lim_{n\to\infty}\left(\frac{1}{1+\frac{r}{n}}\right)^n = e^r$$

- So in continuous time, we use the discount rate

$$\beta = e^{-rt}$$

# The Current Value Hamiltonian

- In problems with discounting, the Hamiltonian is of the form

$$H(x_t, u_t, t) = e^{-\rho t} F(x_t, u_t, t) + \lambda_t A(x_t, u_t, t)$$

- It is to redefine this via a current-value Hamiltonian, $H^c = He^{\rho t}$. Also define $\mu_t = \lambda_t e^{\rho t}$ so that and the current value Hamiltonian is

$$H^c = F(x_t, u_t, t) + \mu_t A(x_t, u_t, t)$$

- Then we can derive that

$$\begin{aligned} \lambda_t &= \mu_t e^{-\rho t} \\ \Longrightarrow \dot{\lambda} &= \frac{d}{dt}\left[\mu_t e^{-\rho t}\right] = \dot{\mu} e^{-\rho t} - \rho \mu_t e^{-\rho t} \end{aligned}$$

- We can use this condition to re-state the maximum principle conditions in terms of the current value Hamiltonian.

# The Maximum Principle for the Current Value Hamiltonian

- Given the current-value Hamiltonian

$$H^c = F(x_t, u_t, t) + \mu_t A(x_t, u_t, t)$$

- Then the following set of conditions will make the path $(x_t^*, u_t^*, \mu_t^*)$ represent an optimal solution:

$$\begin{aligned}
\frac{\partial H^c(x_t^*, u_t^*, \mu_t^*)}{\partial u} &= 0 \\
\frac{\partial H^c(x_t^*, u_t^*, \mu_t^*)}{\partial x} &= -\dot{\mu} + \rho\mu \\
\frac{\partial H^c(x_t^*, u_t^*, \mu_t^*)}{\partial \mu} &= \dot{x} \\
\mu_{t_1} e^{-\rho t_1} &= 0 \Rightarrow \mu_{t_1} = 0
\end{aligned}$$

# Maximum Principle Technicalities

- **Infinite Horizon**: Lots of problems in macroeconomics set $t_1$ to infinity. You might imagine the transversality condition generalises to $\lim_{t\to\infty} \lambda_t = 0$ or $\lim_{t\to\infty} e^{-\rho t}\mu_t = 0$ in the case of the current value Hamiltonian. In fact this doesn't always work. There are cleverly constructed counter-examples. But for the models we will look at this condition will suffice.
- **Second-Order Conditions**: Differentiating and setting equal to zero can yield a maximum, a minimum or a saddle point in $x_t, u_t$ space. We will have obtained a maximum if the Hamiltonian is concave in $(x_t, u_t)$. In fact a weaker condition exists.
- **Arrow's Theorem**: $H(x_t, u_t^*, t)$ is a concave function in $x_t$ then the conditions in the maximum principle are sufficient to ensure a global maximum.
- So if $F$ and $A$ are concave in $x_t$ then we have a global maximum. This condition holds in most economics applications of the maximum principle.

# Part II

## The Ramsey Model

# The Ramsey Model

- As an application of the maximum principle, we will analyse the classic model of optimal consumption and capital known as the Ramsey model in honour of his 1928 article "A Mathematical Theory of Saving." Further developed in the 1960s by Cass and Koopmans so the model has lots of names.
- A social planner is seeking to maximise

$$\int_0^\infty U\left(C_t\right) e^{-\rho t} dt$$

subject to

$$\dot{K}_t = f\left(K_t\right) - C_t - \delta K_t$$

where $f'\left(K_t\right) > 0$, $f''\left(K_t\right) < 0$ and it is assumed that $K_0$ takes some given value.

# Optimality Conditions

- The current value Hamiltonian for this problem is

$$H^c = U\left(C_t\right) + \mu_t\left(f\left(K_t\right) - C_t - \delta K_t\right)$$

- The first-order conditions are

$$\begin{aligned}
\frac{\partial H^c}{\partial C} &= U'\left(C_t\right) - \mu_t = 0 \\
\frac{\partial H^c}{\partial K} &= \mu_t f'\left(K_t\right) - \mu_t \delta = -\dot{\mu}_t + \rho\mu_t \\
\frac{\partial H^c}{\partial \mu} &= f\left(K_t\right) - C_t - \delta K_t = \dot{K}_t \\
\lim_{t\to\infty} \mu_t e^{-\rho t} &= 0
\end{aligned}$$

- We can convert the first three equations into two differential equations in $C_t$ and $K_t$ starting by noting that

$$\frac{\dot{\mu}_t}{\mu_t} = \rho + \delta - f'\left(K_t\right)$$

# Two Differential Equations

- We can use the fact that $\mu_t = U'(C_t)$ to get

$$\dot{\mu}_t = U''(C_t)\,\dot{C}_t$$

  and

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{U''(C_t)\,\dot{C}_t}{U'(C_t)}$$

- So

$$\frac{U''(C_t)\,\dot{C}_t}{U'(C_t)} = \rho + \delta - f'(K_t)$$

- This means we have a system of two nonlinear differential equations describing the dynamics of consumption and capital.

$$\begin{aligned}
\dot{C}_t &= -\left(\frac{U'(C_t)}{U''(C_t)}\right)\left(f'(K_t) - \rho - \delta\right) \\
\dot{K}_t &= f(K_t) - C_t - \delta K_t
\end{aligned}$$

# CRRA Preferences and Consumption Dynamics

- We need to specify a utility function to get a specific solution. A popular option is Constant Relative Risk Aversion (CRRA) utility.

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$$

- This implies a system of differential equations of the form:

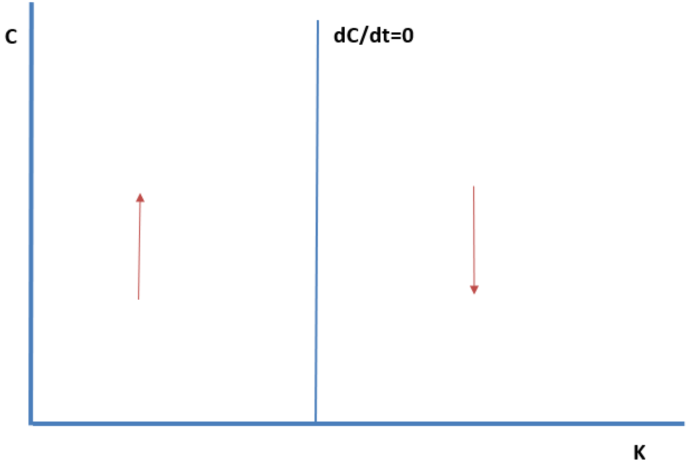
$$\begin{aligned} \frac{\dot{C}_t}{C_t} &= \frac{f'(K_t) - \rho - \delta}{\sigma} \\ \dot{K}_t &= f(K_t) - C_t - \delta K_t \end{aligned}$$

- Now we can start to figure out how to represent this system as a phase diagram. Start with consumption. $\dot{C}_t = 0$ implies

$$f'(K_t) = \rho + \delta$$

Because $f'$ is a monotonically declining function, this means $\dot{C}_t = 0$ is associated with a specific $K^*$ such that $f'(K^*) = \rho + \delta$. For $K_t < K^*$ consumption is increasing. For $K_t > K^*$ consumption is declining.

# The $\frac{dC}{dt} = 0$ Locus



# Capital Dynamics

- The dynamics of the capital stock are given by

$$\dot{K}_t = f(K_t) - C_t - \delta K_t$$

- $\dot{K}_t = 0$ implies

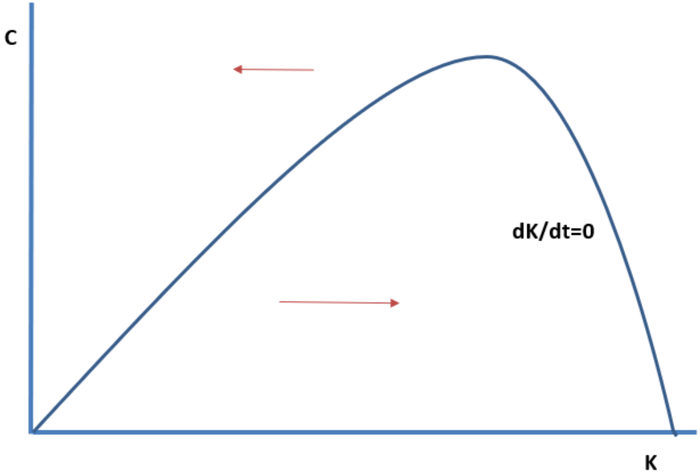
$$C_t = f(K_t) - \delta K_t$$

- The slope of this curve is given by

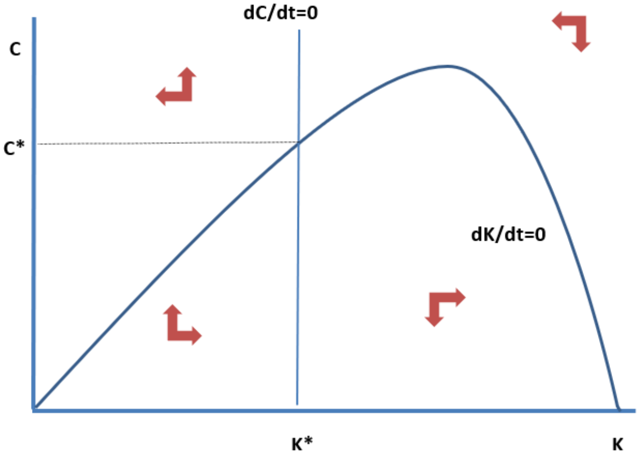
$$\frac{dC_t}{dK_t} = f'(K_t) - \delta$$

- Because $f'(K_t) > 0$ and $f''(K_t) < 0$, the slope of curve will start out positive and then become negative. In other words, the $\dot{K}_t = 0$ starts out rising as $K_t$ increases, reaches a peak and then declines.
- For each value of $K_t$, every point above this curve corresponds to higher consumption. Because $\dot{K}_t$ depends negatively on consumption, capital is declining above the $\dot{K}_t = 0$ line and increasing below it.

# The $\frac{dK}{dt} = 0$ Locus



# The Full System



# The Golden Rule and the Equilibrium Capital Stock

- Define the **golden rule** level of capital, $K^g$, as that level of capital that facilitates the highest possible equilibrium level of consumption.
- We know that along the curve $\dot{K}_t = 0$, we have

$$C_t = f\left(K_t\right) - \delta K_t$$

- Taking the derivative of this with respect to $K_t$

$$\frac{dC_t}{dK_t} = f'\left(K_t\right) - \delta \Rightarrow f'\left(K^g\right) - \delta$$

- Our equilibrium $\left(C^*, K^*\right)$ however, features

$$f'\left(K^*\right) = \rho + \delta$$

- Because there $f''$, this higher marginal productivity of capital implies the equilibrium level of capital in this model is lower than the golden rule level. This also implies a lower level of consumption than the golden rule level.
- This occurs because of impatience due to discounting. We will discuss the economics behind this in a bit.

# Linearised Local Dynamics Around the Equilibrium

- Let's write our system as

$$\begin{aligned}
\dot{C}_t &= C_t \left[ \frac{f'(K_t) - \rho - \delta}{\sigma} \right] = g_1(C_t, K_t) \\
\dot{K}_t &= f(K_t) - C_t - \delta K_t = g_2(C_t, K_t)
\end{aligned}$$

- We approximate these dynamics with a first-order Taylor series approximation around the point $(C^*, K^*)$.

$$g_1(C_t, K_t) = g_1(C^*, K^*) + \frac{\partial g_1(C^*, K^*)}{\partial C}(C_t - C^*) + \frac{\partial g_1(C^*, K^*)}{\partial K}(K_t - K^*)$$

$$g_2(C_t, K_t) = g_2(C^*, K^*) + \frac{\partial g_2(C^*, K^*)}{\partial C}(C_t - C^*) + \frac{\partial g_2(C^*, K^*)}{\partial K}(K_t - K^*)$$

# Linearised Local Dynamics: Consumption

- Start with consumption.

$$g_1(C_t, K_t) = g_1(C^*, K^*) + \frac{\partial g_1(C^*, K^*)}{\partial C}(C_t - C^*) + \frac{\partial g_1(C^*, K^*)}{\partial K}(K_t - K^*)$$

- Calculate the coefficients as follows:

$$\begin{aligned} g_1(C^*, K^*) &= 0 \\ \frac{\partial g_1(C^*, K^*)}{\partial C_t} &= \frac{f'(K^*) - \rho - \delta}{\sigma} = 0 \\ \frac{\partial g_1(C^*, K^*)}{\partial K_t} &= \sigma^{-1} C^* f''(K^*) \end{aligned}$$

- Writing

$$\beta = -\sigma^{-1} C^* f''(K^*)$$

and noting $\beta$ must be positive, we have derived the local dynamics of consumption as

$$\dot{C}_t = -\beta(K_t - K^*)$$

# Linearised Local Dynamics: Capital

- Now calculate the linearised dynamics of the capital stock in a similar fashion:

$$g_2\left(C_t, K_t\right) = g_2\left(C^*, K^*\right) + \frac{\partial g_2\left(C^*, K^*\right)}{\partial C}\left(C_t - C^*\right) + \frac{\partial g_2\left(C^*, K^*\right)}{\partial K}\left(K_t - K^*\right)$$

and the coefficients are given by

$$\begin{aligned} g_2\left(C^*, K^*\right) &= 0 \\ \frac{\partial g_2\left(C^*, K^*\right)}{\partial C_t} &= -1 \\ \frac{\partial g_2\left(C^*, K^*\right)}{\partial K_t} &= f'\left(K^*\right) - \delta = \rho \end{aligned}$$

For the last step remember that $f'\left(K^*\right) = \rho + \delta$.

- The local linearised dynamics for capital are thus

$$\dot{K}_t = -\left(C_t - C^*\right) + \rho\left(K_t - K^*\right)$$

# The Full Linearised System

- The full linearised system is

$$\begin{pmatrix} \dot{C}_t \\ \dot{K}_t \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ -1 & \rho \end{pmatrix} \begin{pmatrix} C_t - C^* \\ K_t - K^* \end{pmatrix}$$

- We calculate its eigenvalues as

$$\begin{vmatrix} -\lambda & -\beta \\ -1 & -\rho - \lambda \end{vmatrix} = 0$$

- Which implies

$$(-\lambda)(\rho - \lambda) - \beta = \lambda^2 - \rho\lambda - \beta = 0$$

  which has roots

$$\lambda_{1,2} = \rho \pm \sqrt{\rho^2 + 4\beta}$$

- This gives two real roots, one positive, one negative. Ruling out the positive root because it generates explosive values, there exists a saddle path along which both consumption and capital move steadily towards $(C^*, K^*)$. Given any initial value of the capital stock, the economy will move towards equilibrium if the social planner chooses the level of consumption required to place the economy on this saddle path.

# Stability of the Equilibrium?

- This model has an equilibrium level of consumption and capital $(C^*, K^*)$ but it is immediately clear from the phase diagram that it is not globally stable.
- We have established there is a local convergent saddle path around $(C^*, K^*)$ but the logic of the phase diagram tells us that there is indeed a unique global saddle path i.e. for any value of the capital stock, there is a unique value of consumption that sets the economy on the path to equilibrium.
- Can we be sure the economy will always be either on the saddle path or in equilibrium? Yes.
- For all the other points, you can show the capital stock heads for either zero or infinity. It turns out, however, that we can rule out both possibilities as not being consistent with the optimality conditions.

# Ruling Out Non-Stable Equilibrium

We can rule out both $K_t \to 0$ and $K_t \to \infty$ as follows.

1. $K_t \to 0$: Capital heads towards zero in the top-left-hand side of the phase diagram and the model obeys the first-order conditions with consumption rising. However, once we hit $K_t = 0$, there can be no production and thus no consumption. This jump downwards in consumption is not consistent with the optimality conditions.

2. $K_t \to \infty$: In this case, we have $f'(K_t) \to 0$. Thus, we have

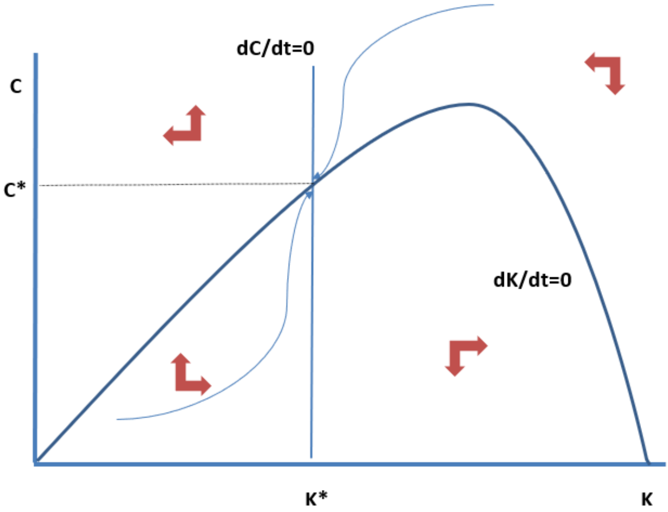
$$\frac{\dot{\mu}_t}{\mu_t} = \rho + \delta + f'(K_t) \to \rho + \delta$$

So the growth rate of $\mu$ tends towards $\rho + \delta$. This means that

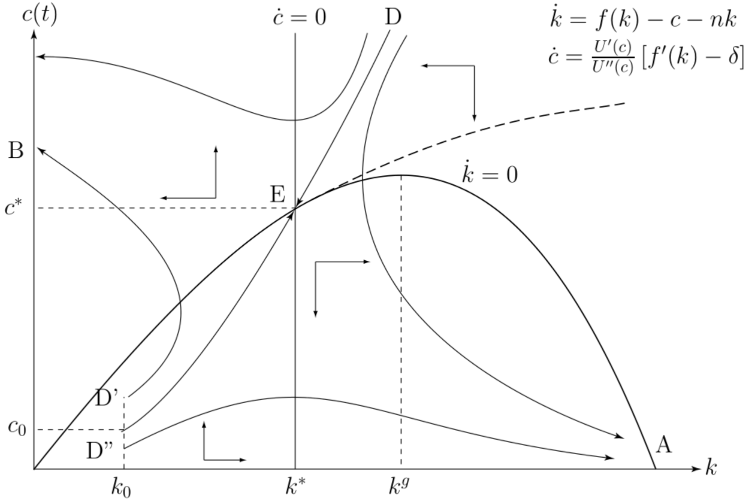
$$\lim_{n \to \infty} \mu_t e^{-\rho t} \neq 0$$

which violates the transversality condition.

# Saddle Path Convergence in the Ramsey Model



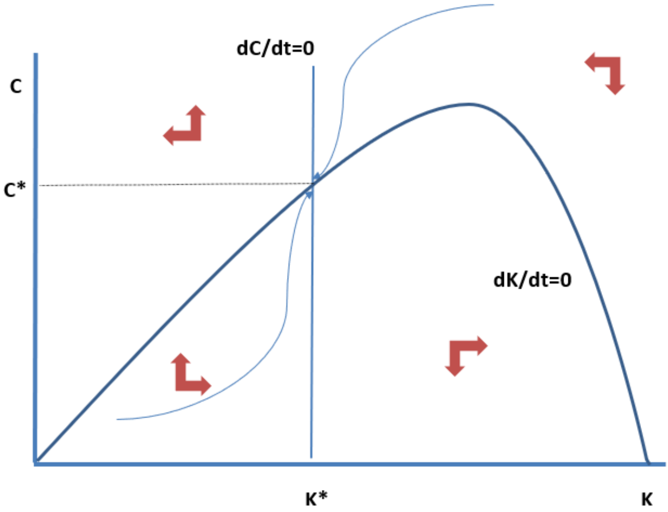
# Other People Make Fancier Ramsey Model Phase Diagram Graphs



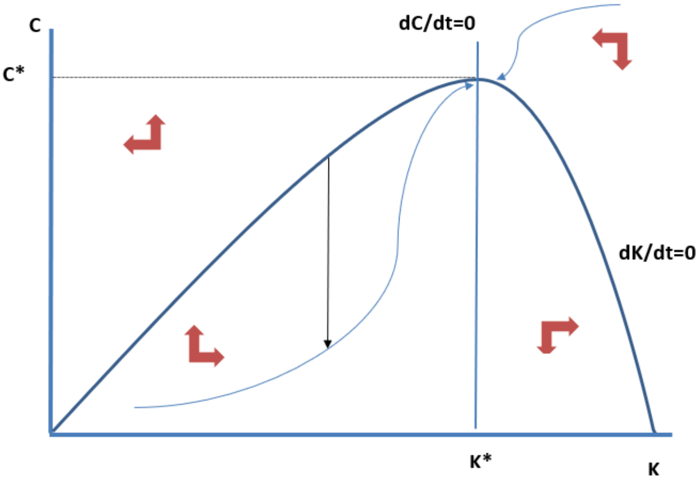
# Why Not Choose Higher Consumption and Capital?

- We have shown that the equilibrium level of consumption and capital, $(C^*, K^*)$ is below the maximum sustainable levels associated with the golden rule. In a sense, this outcome seems suboptimal, so why is this the outcome?
- Consider the following scenario: Starting out from $(C^*, K^*)$, there is a change in preferences and suddenly there is no more time discounting ($\rho = 0$).
- What happens? We know the final equilibrium outcome involves higher consumption and capital. But the shorter run dynamics involve temporarily lower levels of consumption.
- Why is this? The saddle path approaches the equilibrium from below: Building up the extra capital required to deliver the golden rule level of consumption requires extra saving prior to reaching that point, so the dynamics of approaching an equilibrium at $K^g$ starting out from our original $K^*$ require consumption to jump down onto the new saddle path, which lies below the previous saddle path.
- Once our agents have discounting, they don't choose this path to higher capital and consumption because they are not willing to put up with the temporarily lower consumption that you get on the saddle path along the way.

# With Discounting



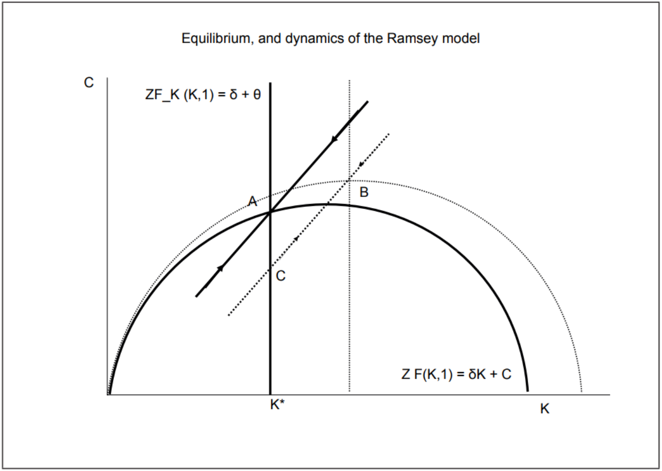
# After a Shock to Set the Discount Rate to Zero



# A Productivity Shock

- You can also do other thought experiments changing other parameters. In each case, what happens is there is a jump in consumption to place the model on its new equilibrium saddle path.
- For example, we can write the production function as $f(K_t) = Ah(K_t)$ and consider a jump in the technology variable $A$.
- While a positive productivity shock definitely leads to higher consumption in the end, in the short run it could either lead to an increase or a decrease in consumption, depending on preferences and technology and the size of the productivity shock.
- See the picture on the next few pages borrowed from Olivier Blanchard's MIT lecture notes.

# A Positive Productivity Shock Might Decrease Consumption At First

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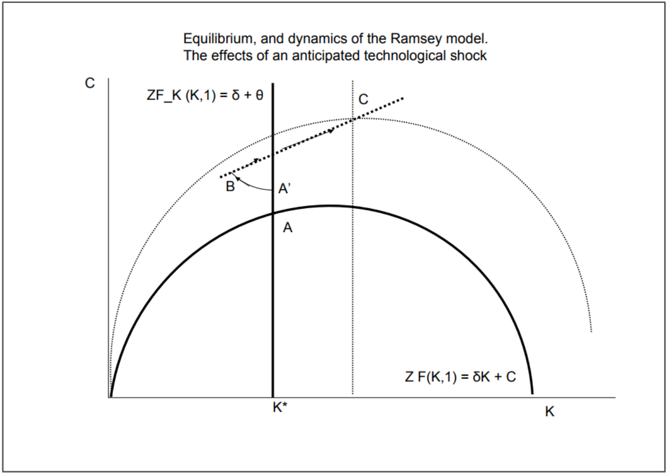
# Or Increase Consumption At First

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Cite as: Olivier Blanchard, course materials for 14.452 Macroeconomic Theory II, Spring 2007.
MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

# Ramsey Model: Lots of Other Extensions

- There are lots of ways to extend the Ramsey model to incorporate other aspects of the economy.
  1. Introducing exogenous technological change.
  2. Separate modelling of households and firms, including potential for imperfect competition.
  3. Open economy modelling including dynamics of exchange rates and current accounts.
  4. Introducing demographics: Births and deaths instead of infinitely-lived agents.
  5. Fiscal policy: Introducing governments and taxes.
  6. Adding investment adjustment costs
  7. Adding exhaustible or renewable resources.
- Lots of these extensions can be found in the classic Blanchard and Fischer textbook.

Part III

# Numerically Calculating the Saddle Path

# Simulating the Ramsey Model

- We have already worked on simulating models on the computer using Matlab.
- When simulating the Ramsey model, there is a complication. We don't want to simulate all the non-equilibrium exploding or collapsing paths. We want to simulate the convergent saddle path, starting from some initial point in time and moving forward.
- For each starting value of capital, there is a unique level of consumption that puts the model on the saddle path. Every other value seems the model explode or collapse. But we don't have any analytical calculations for what these unique set of consumption points are.
- One method is to just analyse the linearised model and set the initial condition for consumption so the coefficient on the unstable eigenvalue is zero and what is left is the convergent saddle path.
- Instead, we're going to use a solution method that will work for the non-linearised method. This method is known as a "shooting algorithm" but that's just a fancy term for "'guessing the initial value of consumption until you get it right."

# A Shooting Algorithm

- Suppose we start out with a low value of initial capital. What happens next?
- Go back and look at our phase diagram for the Ramsey model on page 29.
  - If consumption ever goes above $C^*$, capital ends up going to zero.
  - If capital ever goes above $K^*$, consumption ends up going to zero.
- Our algorithm is based on picking a first-period value for consumption (given our low first-period level of capital), simulating the model (using the Euler finite-difference method) and then updating this initial guess as follows:
  - If consumption ever goes above $C^*$ on the first guess, change the guess for the first-period value of consumption to be half the initial guess, then change subsequent guesses to be an average of the previous two guesses.
  - If capital ever goes above $K^*$ on the first guess, change the guess for the first-period value of consumption to be a weighted average of the initial guess and $C^*$, then change subsequent guesses to be an average of the previous two guesses.
  - Keep going until, after a while, your initial guess delivers a path where $C_t$ converges to $C^*$ after a while.
- A similar approach can be used to figure out convergence paths starting from a high level of capital. It's brute force but it works.

# Programme Set Up

```
%%
% 1. Preliminaries
tic;
clear all; clc; close all; set(0,'DefaultLegendAutoUpdate','off');

% Set latex as default interpreter
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

T    = 100;    % Length of simulation
dt   = 0.2;    % Time step
time = linspace(1,T,T*dt);
crit   = 1;
eps    = 0.005;
iter   = 0;

% Initialising Variables
K = zeros(1,2*T);
C = zeros(1,2*T);
Y = zeros(1,2*T);
dK = zeros(1,2*T-1);
dC = zeros(1,2*T-1);

% Parameters
rho   = 0.99;
alpha = 0.33;
delta = 0.05;
sigma = 5;
kss   = (alpha / (rho+delta))^(1 / (1-alpha) );
css   = (kss)^alpha - delta*kss;
disp('Steady-State Capital Stock');
disp(kss);
disp('Steady-State Consumption');
disp(css);
```

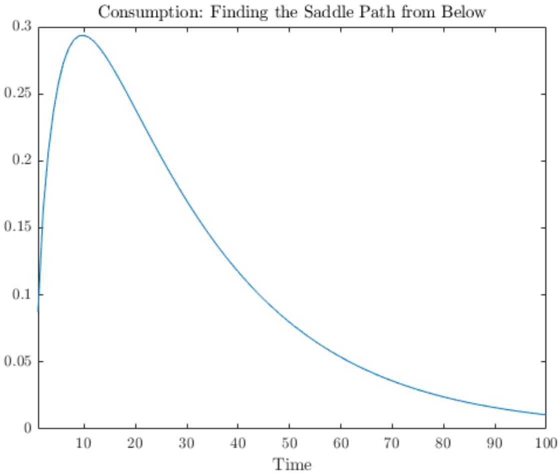
# Calculating a Convergence Path from Below $K^*$

```
% 2. Saddle path from below equilibrium
K(1) = 0.01*kss;
Y(1) = K(1)^alpha;
covershoot = css;
cundershoot = 0;

while iter < 101 & crit > eps
   C(1) = 0.5*(covershoot + cundershoot);

for n=2:T
        C(n) = max(C(n-1) + dt*((1/sigma)*(alpha*K(n-1)^(alpha-1) - rho - delta ))*C(n-1),0) ;
        if C(n) > css
        covershoot = C(1);
        end
        K(n) = max(K(n-1) + dt*(K(n-1)^alpha - C(n-1) - delta*K(n-1) ), 0);
        if K(n) > kss
        cundershoot = C(1);
        end
        Y(n) = K(n)^alpha;

end %for n
 crit = abs( log(C(T)/css) + log(K(T) / kss));
 crit(isnan(crit)) = 1000; % If it returns an NAN, set crit = 1000
 iter = iter+1;

disp('Iteration');
disp(iter);
disp('Crit');
disp(crit);
format long
disp('Initial Consumption');
disp(C(1));
figure(1)
plot(C);
title('Consumption: Finding the Saddle Path from Below');
xlabel('Time')
xlim([1 T])
pause(0.2);

end % while
```

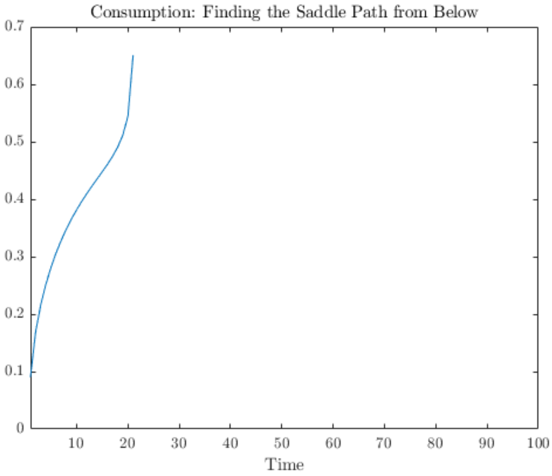
# Repeating the Process When Starting Above $K^*$

```
% 2. Saddle path from below equilibrium
K(1) = 0.01*kss;
Y(1) = K(1)^alpha;
covershoot = css;
cundershoot = 0;

while iter < 101 & crit > eps
   C(1) = 0.5*(covershoot + cundershoot);

for n=2:T
        C(n) = max(C(n-1) + dt*((1/sigma)*(alpha*K(n-1)^(alpha-1) - rho - delta ))*C(n-1),0) ;
        if C(n) > css
        covershoot = C(1);
        end
        K(n) = max(K(n-1) + dt*(K(n-1)^alpha - C(n-1) - delta*K(n-1) ), 0);
        if K(n) > kss
        cundershoot = C(1);
        end
        Y(n) = K(n)^alpha;

end %for n
 crit = abs( log(C(T)/css) + log(K(T) / kss));
 crit(isnan(crit)) = 1000; % If it returns an NAN, set crit = 1000
 iter = iter+1;

disp('Iteration');
disp(iter);
disp('Crit');
disp(crit);
format long
disp('Initial Consumption');
disp(C(1));
figure(1)
plot(C);
title('Consumption: Finding the Saddle Path from Below');
xlabel('Time')
xlim([1 T])
pause(0.2);

end % while
```

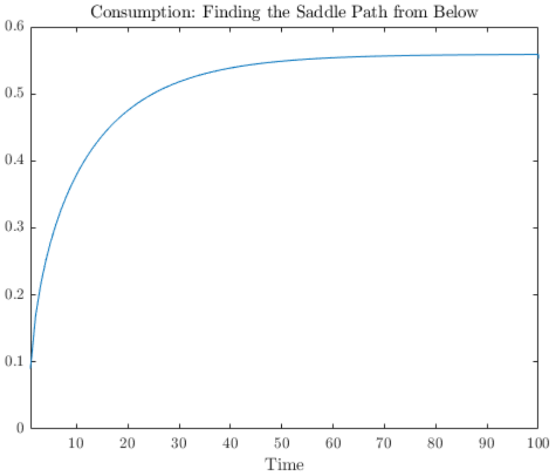
# Initial Guess with Consumption Collapsing



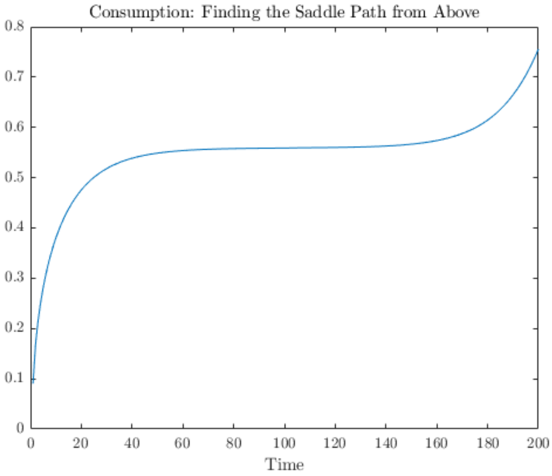
# Initial Guess with Consumption Exploding



# Stop When An Initial Guess Gives Convergence to $C^*$



# The Full Saddle Path



# Using `quiver` to Make a Phase Diagram