PhD Macroeconomics 1:
7. Solving Models with Rational Expectations

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Autumn 2023

Part I

Introduction to Rational Expectations

Introducing Expectations

- VAR models just have backward-looking dynamics but many models in economics also involve people looking forward to the future.
- The backward-looking dynamics stem, for instance, from identities linking today's capital stock with last period's capital stock and this period's investment, i.e. $K_t = (1 - \delta)K_{t-1} + I_t$.
- The forward-looking dynamics stem from optimising behaviour: What agents expect to happen tomorrow is very important for what they decide to do today.
- Modelling this idea requires an assumption about how people formulate expectations.
- In these notes, we will discuss the so-called rational expectations approach to modelling how people think about future events. This is the approach used in so-called Dynamic Stochastic General Equilibrium (DSGE) models.
- In these notes, we introduce the idea of rational expectations and describe how to solve and simulate linear rational expectations models that have both backward and forward-looking components.

Rational Expectations

- Almost all economic transactions rely crucially on the fact that the economy is not a "one-period game." Economic decisions have an **intertemporal** element to them.
- A key issue in macroeconomics is how people formulate expectations about the in the presence of uncertainty.
- Prior to the 1970s, this aspect of macro theory was largely ad hoc. Generally, it was assumed that agents used some simple extrapolative rule whereby the expected future value of a variable was close to some weighted average of its recent past values.
- This approach criticised in the 1970s by economists such as Robert Lucas and Thomas Sargent. Lucas and Sargent instead promoted the use of an alternative approach which they called "rational expectations."
- In economics, rational expectations usually means two things:
  1. They use publicly available information in an efficient manner. Thus, they do not make systematic mistakes when formulating expectations.
  2. They understand the structure of the model economy and base their expectations of variables on this knowledge.

Rational Expectations as a Baseline

- Rational expectations is clearly a strong assumption.
- The structure of the economy is complex and in truth nobody truly knows how everything works.
- But one reason for using rational expectations as a baseline assumption is that once one has specified a particular model of the economy, any other assumption about expectations means that people are making systematic errors, which seems inconsistent with rationality.
- Still, behavioural economists have now found lots of examples of deviations from rationality in people's economic behaviour.
- But rational expectations requires one to be explicit about the full limitations of people's knowledge and exactly what kind of mistakes they make. And while rational expectations is a clear baseline, once one moves away from it there are lots of essentially ad hoc potential alternatives.
- At least at present, the profession has no clear agreed alternative to rational expectations as a baseline assumption.
- And like all models, rational expectations models need to be assessed on the basis of their ability to fit the data.

Part II

Single Stochastic Difference Equations

First-Order Stochastic Difference Equations

- Lots of models in economics take the form

$$y_t = x_t + aE_t y_{t+1}$$

- The equation just says that y today is determined by x and by tomorrow's expected value of y. But what determines this expected value? Rational expectations implies a very specific answer.
- Under rational expectations, the agents in the economy understand the equation and formulate their expectation in a way that is consistent with it:

$$E_t y_{t+1} = E_t x_{t+1} + aE_t E_{t+1} y_{t+2}$$

  This last term can be simplified to

$$E_t y_{t+1} = E_t x_{t+1} + aE_t y_{t+2}$$

  because $E_t E_{t+1} y_{t+2} = E_t y_{t+2}$.
- This is known as the **Law of Iterated Expectations**: It is not rational for me to expect to have a different expectation next period for y_{t+2} than the one that I have today.

Repeated Substitution

- Substituting this into the previous equation, we get

$$y_t = x_t + aE_t x_{t+1} + a^2 E_t y_{t+2}$$

- Repeating this by substituting for $E_t y_{t+2}$, and then $E_t y_{t+3}$ and so on gives

$$y_t = x_t + aE_t x_{t+1} + a^2 E_t x_{t+2} + ... + a^{N-1} E_t x_{t+N-1} + a^N E_t y_{t+N}$$

- Which can be written in more compact form as

$$y_t = \sum_{k=0}^{N-1} a^k E_t x_{t+k} + a^N E_t y_{t+N}$$

- Usually, it is assumed that

$$\lim_{N \to \infty} a^N E_t y_{t+N} = 0$$

- So the solution is

$$y_t = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$

  This solution underlies the logic of a very large amount of modern macroeconomics.

Example: Asset Pricing

- Consider an asset that can be purchased today for price P_t and which yields a dividend of D_t. Suppose there is a close alternative to this asset that will yield a guaranteed rate of return of r.
- Then, for a risk neutral investor will only hold the asset if it yields the same rate of return, i.e. if

$$\frac{D_t + E_t P_{t+1}}{P_t} = 1 + r$$

- This can be re-arranged to give

$$P_t = \frac{D_t}{1+r} + \frac{E_t P_{t+1}}{1+r}$$

- The repeated substitution solution is

$$P_t = \sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^{k+1} E_t D_{t+k}$$

- This equation, which states that asset prices should equal a discounted present-value sum of expected future dividends, is usually known as the **dividend-discount model**.

"Backward" Solutions

- The model

$$y_t = x_t + aE_t y_{t+1}$$

can also be written as

$$y_t = x_t + ay_{t+1} + a\epsilon_{t+1}$$

where ϵ_{t+1} is a forecast error that cannot be predicted at date t.

- Moving the time subscripts back one period and re-arranging this becomes

$$y_t = a^{-1} y_{t-1} - a^{-1} x_{t-1} - \epsilon_t$$

- This backward-looking equation which can also be solved via repeated substitution to give

$$y_t = -\sum_{k=0}^{\infty} a^{-k} \epsilon_{t-k} - \sum_{k=1}^{\infty} a^{-k} x_{t-k}$$

Choosing Between Forward and Backward Solutions

- The forward and backward solutions are both correct solutions to the first-order stochastic difference equation (as are all linear combinations of them). Which solution we choose to work with depends on the value of the parameter a.
- If $|a| > 1$, then the weights on future values of x_t in the forward solution will explode. In this case, it is most likely that the forward solution will not converge to a finite sum. Even if it does, the idea that today's value of y_t depends more on values of x_t far in the distant future than it does on today's values is not one that we would be comfortable with. In this case, practical applications should focus on the backwards solution.
- However, the equation holds for any set of shocks ϵ_t such that $E_{t-1}\epsilon_t = 0$. So the solution is indeterminate: You could not predict what would happen with y_t even if we know the full path for x_t.
- But if $|a| < 1$ then the weights in the backwards solution are explosive and the forward solution is the one to focus on. Also, this solution is determinate. Knowing the path of x_t will tell you the path of y_t.

Rational Bubbles

- In most cases, it is assumed that $|a| < 1$.
- In this case, the assumption that

$$\lim_{N \to \infty} a^N E_t y_{t+N} = 0$$

  amounts to a statement that y_t can't grow too fast.

- What if it doesn't hold? Then the solution can have other elements.
- Let

$$y_t^* = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$

- And let $y_t = y_t^* + b_t$ be any other solution. The solution must satisfy

$$y_t^* + b_t = x_t + aE_t y_{t+1}^* + aE_t b_{t+1}$$

- By construction, one can show that $y_t^* = x_t + aE_t y_{t+1}^*$.

Rational Bubbles, Continued

- This means the additional component satisfies

$$b_t = aE_t b_{t+1}$$

- Because $|a| < 1$, this means b is always expected to get bigger in absolute value, going to infinity in expectation. This is a **bubble**.
- Note that the term bubbles is usually associated with irrational behaviour by investors. But, in this model, the agents have rational expectations. This is a rational bubble.
- There may be restrictions in the real economy that stop b growing forever. But constant growth is not the only way to satisfy $b_t = aE_t b_{t+1}$. The following process also works:

$$b_{t+1} = \begin{cases} (aq)^{-1} b_t + e_{t+1} & \text{with probability} \quad q \\ e_{t+1} & \text{with probability} \quad 1-q \end{cases}$$

where $E_t e_{t+1} = 0$.

- This is a bubble that everyone knows is going to crash eventually. And even then, a new bubble can get going. Imposing $\lim_{N\to\infty} a^N E_t y_{t+N} = 0$ rules out bubbles of this (or any other) form.

From Structural to Reduced Form Relationships

- The solution

$$y_t = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$

  provides useful insights into how the variable y_t is determined.

- However, without some assumptions about how x_t evolves over time, it cannot be used to give precise predictions about they dynamics of y_t.
- Ideally, we want to be able to simulate the behaviour of y_t on the computer.
- One reason there is a strong linkage between DSGE modelling and VARs is that this question is usually addressed by assuming that the exogenous "driving variables" such as x_t are generated by backward-looking time series models like VARs.
- Consider for instance the case where the process driving x_t is

$$x_t = \rho x_{t-1} + \epsilon_t$$

  where $|\rho| < 1$.

From Structural to Reduced Form Relationships, Continued

- In this case, we have
$$E_t x_{t+k} = \rho^k x_t$$
- Now the model's solution can be written as
$$y_t = \left[\sum_{k=0}^{\infty} (a\rho)^k \right] x_t$$
- Because $|a\rho| < 1$, the infinite sum converges to
$$\sum_{k=0}^{\infty} (a\rho)^k = \frac{1}{1 - a\rho}$$
Remember this identity from the famous Keynesian multiplier formula.
- So, in this case, the model solution is
$$y_t = \frac{1}{1 - a\rho} x_t$$
- Macroeconomists call this a **reduced-form** solution for the model: Together with the equation describing the evolution of x_t, it can easily be simulated on a computer.

The Recipe For Simulating Rational Expectations Models

- While this example is obviously a relatively simple one, it illustrates the general principal for getting predictions from rational expectations models:
  1. Obtain **structural** equations involving expectations of future driving variables, (in this case the $E_t x_{t+k}$ terms).
  2. Make assumptions about the time series process for the **driving variables** (in this case x_t)
  3. Solve for a **reduced-form** solution than can be simulated on the computer along with the driving variables.

- Finally, note that the reduced-form of this model also has a VAR-like representation, which can be shown as follows:

$$\begin{aligned} y_t &= \frac{1}{1-a\rho}\left(\rho x_{t-1}+\epsilon_t\right) \\ &= \rho y_{t-1}+\frac{1}{1-a\rho}\epsilon_t \end{aligned}$$

So both the x_t and y_t series have purely backward-looking representations. Even this simple model helps to explain how theoretical models tend to predict that the data can be described well using a VAR.

Second-Order Stochastic Difference Equations

- Variables that are characterized by

$$y_t = \sum_{k=0}^{\infty} a^k E_t x_{t+k}$$

  are **jump variables**. They only depends on what's happening today and what's expected to happen tomorrow. If expectations about the future change, they will jump. Nothing that happened in the past will restrict their movement.

- This may be an ok characterization of financial variables like stock prices but it's harder to argue with it as a description of variables in the real economy like employment, consumption or investment.

- Many models in macroeconomics feature variables which depend on **both** the expected future values and their past values. They are characterized by second-order difference equations of the form

$$y_t = ay_{t-1} + bE_t y_{t+1} + x_t$$

Intuition on Solving Stochastic Difference Equations

- If we have a process of the form of the form

$$y_t = ay_{t-1} + bE_t y_{t+1} + x_t$$

  then it's clear the solution won't be purely forward-looking. The value y_{t-1} will have to be part of the solution.

- We get a solution by coming up with a value of $E_t y_{t+1}$ that is consistent with the underlying process.
- We should expect that this expectation will be a function of expected future values of x_t but we should also expect that they depend on the "initial condition" value of y_{t-1}.
- It turns out this is the case. When there is a unique stable equilibrium, it takes the form of

$$y_t = \rho y_{t-1} + \mu \sum_{k=0}^{\infty} \theta^k E_t x_{t+k}$$

- Note that, in this "structural" solution, the coefficient ρ on y_{t-1} is different (usually larger) than the original coefficient a.

Solving Second-Order Stochastic Difference Equations

- Here's one way of solving second-order SDEs. Suppose there was a value λ such that

$$v_t = y_t - \lambda y_{t-1}$$

followed a first-order stochastic difference equation of the form

$$v_t = \alpha E_t v_{t+1} + \beta x_t$$

We'd know how to solve that for v_t and then back out the values for y_t.

- From the fact that $y_t = v_t + \lambda y_{t-1}$, we can re-write the original equation as

$$\begin{aligned} v_t + \lambda y_{t-1} &= a y_{t-1} + b\left(E_t v_{t+1} + \lambda y_t\right) + x_t \\ &= a y_{t-1} + b E_t v_{t+1} + b\lambda\left(v_t + \lambda y_{t-1}\right) + x_t \end{aligned}$$

- This re-arranges to give

$$(1 - b\lambda) v_t = b E_t v_{t+1} + x_t + \left(b\lambda^2 - \lambda + a\right) y_{t-1}$$

Solving Second-Order Stochastic Difference Equations

- By definition, λ was a number such that the v_t it defined followed a first-order stochastic difference equation. This means that λ satisfies:

$$b\lambda^2 - \lambda + a = 0$$

- This is a quadratic equation, so there are two values of λ that satisfy it. For either of these values, we can characterize v_t by

$$\begin{aligned} v_t &= \frac{b}{1-b\lambda} E_t v_{t+1} + \frac{1}{1-b\lambda} x_t \\ &= \frac{1}{1-b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1-b\lambda} \right)^k E_t x_{t+k} \end{aligned}$$

- And y_t obeys

$$y_t = \lambda y_{t-1} + \frac{1}{1-b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1-b\lambda} \right)^k E_t x_{t+k}$$

- Usually, only one of the potential values of λ is less than one in absolute value, so this delivers the unique stable solution.

Lag Operators

- The lag operator turns a variable dated time t into a variable dated time $t-1$:

$$L y_t = y_{t-1}$$

- Lag operators can be multiplied and added just like normal variables. So, for instance, one can write

$$L^k y_t = y_{t-k}$$

- The forward operator has the reverse effect of the lag operator

$$F^k y_t = y_{t+k}$$

- Lag and forward operators also obey a form of the geometric sum formula. Recall that for $-1 < \beta < 1$, we have

$$\sum_{m=0}^{\infty} \beta^m = \frac{1}{1-\beta}$$

- Recall also that if $-1 < \beta < 1$ and $y_t = \beta E_t y_{t+1} + x_t$ then the solution is

$$y_t = \sum_{m=0}^{\infty} \beta^m E_t x_{t+k}$$

Lag Operators

- The equation

$$y_t = \beta E_t y_{t+1} + x_t$$

can be re-written as

$$y_t = E_t \left[\frac{1}{1 - \beta F} x_t \right]$$

- So we have

$$\frac{1}{1 - \beta F} = \sum_{m=0}^{\infty} \beta^m F^m$$

- The same applies for lag operators

$$\frac{1}{1 - \beta L} = \sum_{m=0}^{\infty} \beta^m L^m$$

- To verify that this is the case, note that if

$$y_t = \beta y_{t-1} + x_t$$

then one can apply repeated substitution to re-write this as

$$y_t = x_t + \beta x_{t-1} + \beta^2 x_{t-2} + \beta^3 x_{t-3} + ...$$

The General Case: Characteristics of a Stable Solution

- Consider the general case of an $(n+m)$-th order stochastic difference equation with m lags and n leads.

$$a_n E_t y_{t+n} + a_{n-1} E_t y_{t+n-1} + ... + a_0 y_t + a_{-1} y_{t-1} + a_{-2} y_{t-2} .. + a_{-m} y_{t-m} = x_t$$

- In many cases, there is no unique stable solution. When there is a unique stable solution, it occurs if m of the roots of the **characteristic equation**

$$a_n \lambda^{n+m} + a_{n-1} \lambda^{n+m-1} + ... + a_0 \lambda^m + a_{-1} \lambda^{m-1} + a_{-2} \lambda^{m-2} .. + a_{-m} = 0$$

are inside the unit circle while the other n roots are outside the unit circle.

- If there is a unique stable solution, we can re-write the underlying model as

$$\left(b_0 + b_1 L + b_2 L^2 + ... + b_m L^m\right)\left(c_0 + c_1 F + c_2 F^2 + ... + c_n F^n\right) y_t = x_t$$

where all the roots of the polynomial $b_0 + b_1 L + b_2 L^2 + ... + b_m L^m$ are inside the unit circle and all the roots of the polynomial $c_0 + c_1 F + c_2 F^2 + ... + c_n F^n$ are outside the unit circle.

The General Case: Getting a Solution

- The solution can thus be written

$$\left(b_0 + b_1 L + b_2 L^2 + ... + b_m L^m\right) y_t = \left(c_0 + c_1 F + c2F^2 + ... + c_n F^n\right)^{-1} x_t$$

- The inverse of the polynomial with roots inside the unit circle can be re-expressed as the sum of different terms using the method of partial fractions:

$$\left(c_0 + c_1 F + c_2 F^2 + ... + c_n F^n\right)^{-1} = \sum_{j=1}^{n} \frac{m_j}{1 - \lambda_j F} = \sum_{j=1}^{n} m_j \sum_{k=0}^{\infty} \lambda_j^k E_t x_{t+k}$$

- So the stable solution is of the form

$$\left(b_0 + b_1 L + b_2 L^2 + ... + b_m L^m\right) y_t = \sum_{k=0}^{\infty} \mu_k E_t x_{t+k}$$

where

$$\mu_k = \sum_{j=1}^{n} m_j \lambda_j^k$$

The General Case: Solution with an Autoregressive Driving Variable

- If the "driving variable" x_t is determined by an AR(p) process such that

$$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + ... + \alpha_p x_{t-p} + \epsilon_t$$

  the you can get a "closed form" solution that you can put on a computer and run.

- Hansen and Sargent (*JEDC*, 1980) showed that the expectational term can be re-expressed as a function of current and past values of x_t i.e. that

$$\sum_{k=0}^{\infty} \mu_k E_t x_{t+k} = \sum_{i=0}^{p-1} d_i x_{t-i}$$

  where the d_i coefficients are functions of the μ_k and α_i coefficients.

- So we have a solution of the form

$$y_t = \frac{1}{b_0} \left[-\sum_{i=0}^{m} b_i y_{t-i} + \sum_{i=0}^{k-1} d_i x_{t-i} \right]$$

Part III

Systems of Stochastic Difference Equations

Systems of Rational Expectations Equations

- Suppose one has a vector of variables

$$Z_t = \begin{pmatrix} z_{1t} \\ z_{2t} \\ . \\ z_{nt} \end{pmatrix}$$

- It turns out that most macroeconomic models with rational expectations can be represented by an equation of the form

$$Z_t = BE_t Z_{t+1} + X_t$$

where B is an $n \times n$ matrix. The logic of repeated substitution can also be applied to this model, to give a solution of the form

$$Z_t = \sum_{k=0}^{\infty} B^k E_t X_{t+k}$$

- As with the VAR case we discussed before, this system will produce a stable stationary model if the eigenvalues of B are all inside the unit circle.

Generality of First-Order Matrix Formulation

- Remember how the first-order matrix formulation of the VAR model could be used to represent models with more than one lag?
- At first glance, it looks as the model on the previous slide only allows for first-order purely forward-looking difference equations. However, it turns out that the same "companion matrix" trick can also be applied here, so that this formulation can apply to more general models.
- Consider, for instance, the second order stochastic difference equation

$$y_t = ay_{t-1} + bE_t y_{t+1} + x_t$$

- This model has a forward-looking and backward-looking element. To see that this model can still fit within the first-order matrix formulation, note that it can be re-written as

$$y_{t-1} = \frac{1}{a}y_t - \frac{b}{a}E_t y_{t+1} - \frac{1}{a}x_t$$

- This can be expressed in first-order matrix form as

$$\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b}{a} & \frac{1}{a} \end{pmatrix} E_t \begin{pmatrix} y_{t+1} \\ y_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{a}x_t \end{pmatrix}$$

Link Between Characteristic Equations and Eigenvalues

- Earlier, we discussed how the eigenvalues of a matrix A in VAR analysis is key to whether the VAR is stable or not.
- With stochastic difference equations, we have focused on the roots of the characteristic equation to figure out whether there is stability.
- It turns out that these two are related to each other.
- So, for example, the roots of the characteristic equation of the second-order stochastic difference equation on the previous page are the same as the eigenvalues of the matrix when the single equation is re-expressed in matrix form.

Predetermined Variables

- The idea of expressing rational expectations models in first-order matrix form was first discussed by Blanchard and Kahn (1980).
- One point stressed in their paper is that we do not necessarily want all of the eigenvalues of the B matrix to be less than one.
- Consider the second-order difference equation model just discussed.
- In this case, the Z_t vector contains a term (y_{t-1}) for which a forward-looking solution is not appropriate: y_{t-1} is predetermined (to use the terminology of Blanchard and Kahn) at time t. This means it can't jump up and down with expectations that are determined at time t.
- For a model like this, we would expect B to have eigenvalues both inside and outside the unit circle.

General Solution a la Blanchard and Kahn (1980)

- To provide a more concrete illustration of this idea, let us again consider the model

$$Z_t = BE_t Z_{t+1} + X_t$$

where the matrix B can be written as

$$B = P\Omega P^{-1}$$

where P is a matrix of eigenvectors and Ω is a diagonal matrix with eigenvalues.

- We can re-write the model as

$$Z_t = \left(P\Omega P^{-1}\right) E_t Z_{t+1} + X_t$$

where P is a matrix of eigenvectors.

- Now multiply both sides by P^{-1} to get

$$P^{-1} Z_t = \Omega E_t \left(P^{-1} Z_{t+1}\right) + P^{-1} X_t$$

- Define new vectors of variables

$$\begin{aligned} W_t &= P^{-1} Z_t \\ V_t &= P^{-1} X_t \end{aligned}$$

General Solution a la Blanchard and Kahn (1980)

- The model can be re-written as

$$W_t = \Omega E_t W_{t+1} + V_t$$

- This model is just n separate equations of the form

$$w_{it} = \lambda_i E_t w_{i,t+1} + v_{it}$$

- We can then solve each of these separate equations in the appropriate manner: Those with $|\lambda_i| < 1$ can be solved forward and those with $|\lambda_i| > 1$ can be solved backwards. While backward solutions generally have a multiplicity of solutions, models with pre-determined variables can use specified lagged values of these variables to pin down a unique solution.
- Once solutions for W_t are obtained, we can then obtain solutions for the variables of interest by calculating $Z_t = PW_t$.
- There are various Matlab routines available that can take models of this sort and simulate them and calculate impulse responses.

Getting Reduced-Form Solutions That Can Be Simulated

- In general, systems of rational expectations models that can be written as

$$Z_t = AZ_{t-1} + BE_t Z_{t+1} + X_t$$

have a solution of the form

$$Z_t = CZ_{t-1} + \sum_{k=0}^{\infty} F^k E_t \left(GX_{t+k} \right)$$

where C, F and G are functions of the coefficients in the matrices A and B.

- Now let's assume that the "driving variables" X_t follow a VAR representation of the form

$$X_t = DX_{t-1} + \epsilon_t$$

where D has eigenvalues inside the unit circle.

- The transformed driving variables GX_t also follow a VAR process of the form

$$GX_t = \left(GDG^{-1} \right) \left(GX_{t-1} \right) + G\epsilon_t = R \left(GX_{t-1} \right) + G\epsilon_t$$

where

$$R = \left(GDG^{-1} \right)$$

Getting Reduced-Form Solutions That Can Be Simulated

- This VAR process for the transformed driving variables implies that

$$E_t G X_{t+k} = R^k G X_t$$

- So the model has a solution of the form

$$Z_t = C Z_{t-1} + \left(\sum_{k=0}^{\infty} F^k R^k \right) G X_t$$

- This infinite geometric sum of matrices looks a lot like the "multiplier-like" geometric sum from our earlier example and indeed it is. If the eigenvalues of FR are less than one (as they will be if both F and D have eigenvalues less then one themselves) then this infinite sum converges to

$$\sum_{k=0}^{\infty} F^k R^k = (I - FR)^{-1}$$

- So the model has a reduced-form representation

$$Z_t = C Z_{t-1} + (I - FR)^{-1} G X_t$$

which can be simulated along with the VAR process for the driving variables.