PhD Macroeconomics 1:
8. Examples of Models with Rational Expectations

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Part I

Consumption

A Model of Optimising Consumers

- Here, we will look at the question of how a consumer with rational expectations will plan their spending over a lifetime.
- We will
  1. Show how consumption depends on net wealth and expectations of future income.
  2. Illustrate some pitfalls in using econometrics to assess the effects of policy.
  3. Discuss the impact of tax cuts on consumption spending.
  4. Explain the concept of precautionary savings, which will come up again later in the module.

The Household Budget Constraint

- Let A_t be household assets, Y_t be labour income, and C_t stand for consumption spending. Stock of assets changes by

$$A_{t+1} = (1 + r_{t+1})(A_t + Y_t - C_t)$$

  where r_{t+1} is the return on household assets at time $t + 1$.

- Note that Y_t is **labour** income (income earned from working) not total income because total income also includes the capital income earned on assets (i.e. total income is $Y_t + r_{t+1}A_t$.)

- This can be written as a first-order difference equation in our standard form

$$A_t = C_t - Y_t + \frac{A_{t+1}}{1 + r_{t+1}}$$

- Assume that agents have rational expectations and that return on assets equals a constant, r:

$$A_t = C_t - Y_t + \frac{1}{1 + r} E_t A_{t+1}$$

The Intertemporal Budget Constraint

- We have another first-order stochastic difference equation

$$A_t = C_t - Y_t + \frac{1}{1+r} E_t A_{t+1}$$

- Using the same repeated substitution method as before, we get

$$A_t = \sum_{k=0}^{\infty} \frac{E_t \left(C_{t+k} - Y_{t+k} \right)}{\left(1+r\right)^k}$$

- We are assuming $\frac{E_t A_{t+k}}{(1+r)^k}$ goes to zero as k gets large.
- One way to understand this equation is to re-writing it as

$$\sum_{k=0}^{\infty} \frac{E_t C_{t+k}}{\left(1+r\right)^k} = A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{\left(1+r\right)^k}$$

- This is called the **intertemporal budget constraint**. The present value sum of current and future household consumption must equal the current stock of financial assets plus the present value sum of current and future labour income.

Optimising Consumers

- We will assume that consumers wish to maximize a welfare function of the form

$$W = \sum_{k=0}^{\infty} \left(\frac{1}{1+\beta} \right)^k U\left(C_{t+k}\right)$$

  where $U\left(C_t\right)$ is the instantaneous utility obtained at time t, and β is a positive number that describes the fact that households prefer a unit of consumption today to a unit tomorrow.

- If the future path of labour income is known, consumers choose a path for consumption to maximise the following Lagrangian:

$$L = \sum_{k=0}^{\infty} \left(\frac{1}{1+\beta} \right)^k U\left(C_{t+k}\right) + \lambda \left[A_t + \sum_{k=0}^{\infty} \frac{Y_{t+k}}{(1+r)^k} - \sum_{k=0}^{\infty} \frac{C_{t+k}}{(1+r)^k} \right]$$

- For every current and future value of consumption, C_{t+k}, this yields a first-order condition of the form

$$\left(\frac{1}{1+\beta} \right)^k U'\left(C_{t+k}\right) - \frac{\lambda}{(1+r)^k} = 0$$

Consumption Euler Equation

- For $k = 0$, this implies
$$U'(C_t) = \lambda$$
- For $k = 1$, it implies
$$U'(C_{t+1}) = \left(\frac{1+\beta}{1+r}\right)\lambda$$
- Putting these two equations together, we get
$$U'(C_t) = \left(\frac{1+r}{1+\beta}\right)U'(C_{t+1})$$
- When there is uncertainty about future labour income, this optimality condition can just be re-written as
$$U'(C_t) = \left(\frac{1+r}{1+\beta}\right)E_t\left[U'(C_{t+1})\right]$$

  This implication of the first-order conditions for consumption is sometimes known as an **Euler equation**.

On FOCs When There Is Uncertainty

- Wait a minute, can you really do that last step where you just added an E_t to the uncertain term?
- Well, it's complicated. The formal mathematics of infinite horizon maximisation under uncertainty is complicated. Technically, the best way to solve these problems is using **stochastic dynamic programming** which explains how the optimisation problem is solved on a step-by-step basis. We will cover dynamic programming later in the module.
- For now, let me provide a quick justification for substituting $E_t [U' (C_{t+1})]$ for $U' (C_{t+1})$. Suppose

$$G(x) = \sum_{k=1}^{N} p_k F(a_k, x)$$

- This is maximized by setting

$$G'(x) = \sum_{k=1}^{N} p_k F'(a_k, x) = E_t F'(x) = 0$$

So, the FOCs for for maximizing $E_t F(x)$ are just $E_t F'(x) = 0$.

The Random Walk Theory of Consumption

- In an important 1978 paper, Robert Hall discussed a specific case of the consumption Euler equation. He assumed

$$U(C_t) = aC_t + bC_t^2$$
$$r = \beta$$

- In this case, the Euler equation becomes

$$a + 2bC_t = E_t\left[a + 2bC_{t+1}\right]$$

- Thus which simplifies to

$$C_t = E_t C_{t+1}$$

- Because, the Euler equation holds for all time periods, we have

$$C_t = E_t\left(C_{t+k}\right) \qquad k = 1, 2, 3, ...$$

- All future expected values of consumption equal the current value. Because it implies that changes in consumption are unpredictable, this is sometimes called the **random walk** theory of consumption.

The Rational Expectations Permanent Income Hypothesis

- Consumption changes are unpredictable but what determines the level of consumption each period? Insert $E_t C_{t+k} = C_t$ into the intertemporal budget constraint to get

$$\sum_{k=0}^{\infty} \frac{C_t}{(1+r)^k} = A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k}$$

- Now we can use the geometric sum formula to turn this into a more intuitive formulation:

$$\sum_{k=0}^{\infty} \frac{1}{(1+r)^k} = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r}$$

- So, Hall's assumptions imply the following equation, which we will term the **Rational Expectations Permanent Income Hypothesis**:

$$C_t = \frac{r}{1+r} A_t + \frac{r}{1+r} \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k}$$

Implications of RE-PIH

- The Rational Expectations Permanent Income Hypothesis

$$C_t = \frac{r}{1+r} A_t + \frac{r}{1+r} \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k}$$

states that the current value of consumption is driven by three factors:

1. The expected present discounted sum of current and future labour income.
2. The current value of household assets. This "wealth effect" is likely to be an important channel through which financial markets affect the macroeconomy.
3. The expected return on assets: This determines the coefficient, $\frac{r}{1+r}$, that multiplies both assets and the expected present value of labour income. In this model, an increase in this expected return raises this coefficient, and thus boosts the propensity to consume from wealth.

An Example: Constant Expected Growth in Income

- Suppose households expect labour income to grow at a constant rate g:

$$E_t Y_{t+k} = (1+g)^k Y_t$$

- This implies

$$C_t = \frac{r}{1+r} A_t + \frac{r Y_t}{1+r} \sum_{k=0}^{\infty} \left(\frac{1+g}{1+r} \right)^k$$

- As long as $g < r$ (and we will assume it is) then we can use the geometric sum formula to simplify this expression

$$\sum_{k=0}^{\infty} \left(\frac{1+g}{1+r} \right)^k = \frac{1}{1 - \frac{1+g}{1+r}} = \frac{1+r}{r-g}$$

- This implies a consumption function of the form

$$C_t = \frac{r}{1+r} A_t + \frac{r}{r-g} Y_t$$

- Note that the higher is expected future growth in labour income g, the larger is the coefficient on today's labour income and thus the higher is consumption.

A Warning About Econometrics and Policy Evaluation

- Consider an economy where households have always expected their after-tax labour income to grow at rate g.
- Now suppose the government decide to introduce a one-period income tax cut that boosts after-tax labour income by one unit.
- They ask an econometrician to figure out how much this will raise consumption. The econometrican goes to the data which previously has been characterised by

$$C_t = \frac{r}{1+r} A_t + \frac{r}{r-g} Y_t$$

  and says the answer is $\frac{r}{r-g}$.
- In reality, that relationship only works when people expect labour income growth of g and that won't hold anymore when there is a once-off temporary tax cut. The true model is still

$$C_t = \frac{r}{1+r} A_t + \frac{r}{1+r} \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k}$$

  so consumption will only go up by $\frac{r}{1+r}$.

The Lucas Critique

- How badly does the econometrician get it wrong?
- Suppose $r = 0.06$ and $g = 0.02$. In this case, the economic advisor concludes that the effect of a dollar of tax cuts is an extra 1.5 $(= \frac{.06}{.06-.02})$ dollars of consumption. In reality, the tax cut will produce only an extra 0.057 $(= \frac{.06}{1.06})$ dollars of extra consumption. This is a big difference.
- This may seem like a cooked-up example. But the idea that coefficients in statistical relationships depend upon expectations and that these expectations may change when policy change is not so strange.
- In a famous 1976 paper, Robert Lucas argued that this kind of problem could often lead to econometric analysis providing the wrong answer to various questions about how policy changes would affect the economy.
- This idea that econometric models may be limited in usefulness when analysing policy change (and that it may be better to use theoretically-founded models that incorporate how people formulate expectations) is now known as the **Lucas critique** of econometric policy evaluation.

Explicitly Introducing Fiscal Policy

- Let's change the household budget constraint to explicitly incorporate taxes.
- The household budget constraint is now

$$A_{t+1} = (1+r)(A_t + Y_t - T_t - C_t)$$

  where T_t is taxes paid by the household at time t.
- The household's intertemporal budget constraint becomes

$$\sum_{k=0}^{\infty} \frac{E_t C_{t+k}}{(1+r)^k} = A_t + \sum_{k=0}^{\infty} \frac{E_t (Y_{t+k} - T_{t+k})}{(1+r)^k}$$

- This equation makes it more explicit that households have to factor in all future levels of taxes when making their current spending decisions.

The Government's Budget Constraint

- Like households, governments also have budget constraints.
- The stock of public debt, D_t evolves over time according to

$$D_{t+1} = (1+r)(D_t + G_t - T_t)$$

where G_t is government spending and T_t is tax revenue.

- Applying the repeated-substitution method we can obtain an intertemporal version of the government's budget constraint.

$$\sum_{k=0}^{\infty} \frac{E_t T_{t+k}}{(1+r)^k} = D_t + \sum_{k=0}^{\infty} \frac{E_t G_{t+k}}{(1+r)^k}$$

- This states that the present discounted value of tax revenue must equal the current level of debt plus the present discounted value of government spending.
- In other words, in the long-run, the government must raise enough tax revenue to pay off its current debts as well as its current and future spending.

A Quick Comment on the Government's Budget Constraint

- In deriving the intertemporal version of the government's budget constraint

$$\sum_{k=0}^{\infty} \frac{E_t T_{t+k}}{(1+r)^k} = D_t + \sum_{k=0}^{\infty} \frac{E_t G_{t+k}}{(1+r)^k}$$

  we implicitly assumed that

$$\lim_{k \to \infty} \frac{D_t}{(1+r)^k} = 0$$

- Why might we do that? One reason is that debt may be restricted so that it can't continually grow faster than GDP. Call the GDP growth rate g. If g represents an upper bound on the growth rate for public debt, then the limit above will hold as long as $r > g$.
- However, the r for public debt in advanced countries has been low for a long time. Blanchard (2019) argues that the modern macroconomic environment is characterised by $r < g$ for government debt. This means the intertemporal budget constraint above does not imply—governments don't have to pay off current debt and they can still see debt-GDP ratios stabilise.

Ricardian Equivalence

- Remembering the household intertemporal budget constraint

$$\sum_{k=0}^{\infty} \frac{E_t C_{t+k}}{(1+r)^k} = A_t + \sum_{k=0}^{\infty} \frac{E_t \left(Y_{t+k} - T_{t+k}\right)}{(1+r)^k}$$

- And the governments intertemporal budget constraint

$$\sum_{k=0}^{\infty} \frac{E_t T_{t+k}}{(1+r)^k} = D_t + \sum_{k=0}^{\infty} \frac{E_t G_{t+k}}{(1+r)^k}$$

- The household intertemporal budget constraint becomes

$$\sum_{k=0}^{\infty} \frac{E_t C_{t+k}}{(1+r)^k} = A_t - D_t + \sum_{k=0}^{\infty} \frac{E_t \left(Y_{t+k} - G_{t+k}\right)}{(1+r)^k}$$

- Before, we had discussed how a temporary cut in taxes should have a small effect. This is a more extreme result — unless governments plan to change the profile of government spending, a cut to taxes today has no impact on consumption spending. Households anticipate that lower taxes today will just trigger higher taxes tomorrow.

Evidence on the RE-PIH

- There have been lots of macroeconomic studies on how well the RE-PIH fits the data.
- There are various reasons why the RE-PIH may not hold.
  1. Consumption smoothing may not be possible e.g. banks may not be willing to lend to people on the basis of their expected future income (i.e. there may be "liquidity constraints.")
  2. People may not have rational expectations and may not plan their spending decisions in the calculating optimising fashion assumed by the theory.
- The 1980s saw a large amount of research on whether the RE-PIH fitted the data. The most common conclusion was that consumption was "excessively sensitive" to current disposable income.
- Campbell and Mankiw (1990) introduced a model in which a fraction of the households behave according to the RE-PIH while the rest simply consume all of their current income. They estimate the fraction of non-PIH consumers to be about a half. A common interpretation of this result is that liquidity constraints have an important impact on aggregate consumption.

Evidence on Ricardian Equivalence: Macro

- There is also a large literature devoted to testing the Ricardian equivalence hypothesis. In addition to the reasons the RE-PIH itself may fail, there are other reasons why Ricardian equivalence may not hold.
  1. People don't live forever and so may not worry about future tax increases that could occur in the far future.
  2. Taxes take a more complicated form than the simple lump-sum payments presented above.
  3. The interest rate in the government's budget constraint may not be the same as the interest rate in the household's constraint.
  4. People may often be unable to tell whether tax changes are temporary or permanent.
- Most macro studies find effects of fiscal policy are quite different from the Ricardian equivalence predictions.
- The evidence generally suggests that tax cuts and increases in government spending tend to boost the economy.

Evidence on Ricardian Equivalence: Micro

- Perhaps more interesting are micro-studies of explicitly temporary tax cuts or rebates. These generally find people spend more of the increase in income than the PIH predicts.
  1. Parker et al (2013) studied effects of rebate cheques mailed to households and estimate that people spent between 50 and 90 percent of the rebate in the three-month period after they receive the payment.
  2. Other studies show people increasing spending in response to transitory changes in their social security taxes or once-off tax rebates.
  3. Often the people doing the extra spending are well-off households that are probably not subject to liquidity constraints.
- Still, people don't go on a splurge every time they get a large payment. Hsieh (2003) examines how people in Alaska responded to large anticipated annual payments that they received from a state fund that depended largely on oil revenues. He finds that Alaskan households respond to these payments in line with the predictions of the PIH, smoothing out their consumption over the year.
- And the evidence suggests that a large fraction of the stiumulus cheques given to US households during the Covid pandemic were saved.

Certainty Equivalence

- Let's keep assumption that $r = \beta$, so the Euler equation is
$$U'(C_t) = E_t [U'(C_{t+1})]$$
You might think that this expression is consistent with constant expected consumption but it is not.
- This is because for functions F generally $E(F(X)) \neq F(E(X))$. For concave functions—those with negative second derivatives—a famous result known as Jensen's inequality states that $E(F(X)) < F(E(X))$.
- In this example, we are looking at the properties of $E_t [U'(C_{t+1})]$. Whether or not marginal utility is concave or convex depends on its second derivative, so it depends upon the third derivative of the utility function U'''.
- Most standard utility functions have positive third derivatives implying convex marginal utility and thus $E_t [U'(C_{t+1})] > U'(E_t C_{t+1})$.
- Quadratic utility function was a special case because it has $U''' = 0$, its marginal utility is neither concave or convex and the Jensen relationship is an equality. In this very particular case, the utility function displays **certainty equivalence**: The uncertain outcome is treated the same way as if people were certain of achieving its average value.

Caballero's Example of Non-Certainty Equivalence

- Suppose consumers have a utility function of the form

$$U(C_t) = -\frac{1}{\alpha}\exp\left(-\alpha C_t\right)$$

where exp is the exponential function.

- This implies marginal utility of the form

$$U'\left(C_t\right) = \exp\left(-\alpha C_t\right)$$

- In this case, the Euler equation becomes

$$\exp\left(-\alpha C_t\right) = E_t\left(\exp\left(-\alpha C_{t+1}\right)\right)$$

- Now suppose C_{t+1} is perceived to have a normal distribution with mean $E_t(C_{t+1})$ and variance σ^2. A useful result from statistics is that if a variable X is normally distributed has mean μ and variance σ^2:

$$X \sim N\left(\mu, \sigma^2\right)$$

then one can show that

$$E\left(\exp(X)\right) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

Uncertainty-Induced Tilt

- This result implies that

$$E_t\left(\exp\left(-\alpha C_{t+1}\right)\right) = \exp\left(E_t\left(-\alpha C_{t+1}\right) + \frac{Var\left(-\alpha C_{t+1}\right)}{2}\right)$$
$$= \exp\left(-\alpha E_t\left(C_{t+1}\right) + \frac{\alpha^2\sigma^2}{2}\right)$$

- So, the Euler equation can be written as

$$\exp\left(-\alpha C_t\right) = \exp\left(-\alpha E_t\left(C_{t+1}\right) + \frac{\alpha^2\sigma^2}{2}\right)$$

- Taking logs of both sides this becomes

$$-\alpha C_t = -\alpha E_t\left(C_{t+1}\right) + \frac{\alpha^2\sigma^2}{2}$$

which simplifies to

$$E_t\left(C_{t+1}\right) = C_t + \frac{\alpha\sigma^2}{2}$$

- Even though expected marginal utility is flat, consumption tomorrow is expected to be higher than consumption today.

Precautionary Savings

- Uncertainty induces an "upward tilt" to the consumption profile. And this upward tilt has an affect on today's consumption: We cannot sustain higher consumption tomorrow without having lower consumption today.
- We can calculate exactly what the effect of uncertainty is on consumption today. The Euler equation implies that

$$E_t\left(C_{t+k}\right) = C_t + \frac{k\alpha\sigma^2}{2}$$

- Inserting this into the intertemporal budget constraint, we get

$$\sum_{k=0}^{\infty} \frac{C_t}{(1+r)^k} + \frac{\alpha\sigma^2}{2} \sum_{k=1}^{\infty} \frac{k}{(1+r)^k} = A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k}$$

we can show that

$$\sum_{k=1}^{\infty} \frac{k}{(1+r)^k} = \frac{1+r}{r^2}$$

Precautionary Savings

- So, the intertemporal budget constraint simplifies to

$$\sum_{k=0}^{\infty} \frac{C_t}{(1+r)^k} + \frac{1+r}{r^2}\frac{\alpha\sigma^2}{2} = A_t + \sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k}$$

- Taking the same steps as before, consumption today is

$$C_t = \frac{r}{1+r}A_t + \frac{r}{1+r}\sum_{k=0}^{\infty} \frac{E_t Y_{t+k}}{(1+r)^k} - \frac{\alpha\sigma^2}{2r}$$

- This is exactly as before apart from an additional "precautionary savings" term $-\frac{\alpha\sigma^2}{2r}$. The more uncertainty there is, the more lower the current level of consumption will be.

- This particular result obviously relies on very specific assumptions about the form of the utility function and the distribution of uncertain outcomes. However, since almost all utility function feature positive third derivatives, the key property underlying the precautionary savings result—marginal utility averaged over the uncertain outcomes being higher than at the average level of consumption—will generally hold.

Part II

The New Keynesian Phillips Curve

The Calvo Model of Sticky Prices

- The idea that prices may be "sticky" and that this stickiness make output sensitive to aggregate demand is a central feature of Keynesian economics.
- Here we introduce a model of inflation with a formulation of sticky prices known as **Calvo pricing**, after the economist who first introduced it.
- It is not the most realistic formulation of sticky prices but it provides analytically convenient expressions, and has implications that are very similar to those of more realistic (but more complicated) formulations.
- Only a random fraction $(1-\theta)$ of firms are able to reset their price; all other firms keep their prices unchanged.
- When firms do get to reset their price, they must take into account that the price may be fixed for many periods. We assume they do this by choosing a log-price, z_t, that minimizes the "loss function"

$$L(z_t) = \sum_{k=0}^{\infty} (\theta\beta)^k E_t \left(z_t - p^*_{t+k}\right)^2$$

where β is between zero and one, and p^*_{t+k} is the log of the optimal price that the firm would set in period $t+k$ if there were no price rigidity.

Explaining the Loss Function

$$L(z_t) = \sum_{k=0}^{\infty} (\theta\beta)^k \, E_t \left(z_t - p^*_{t+k}\right)^2$$

- $E_t \left(z_t - p^*_{t+k}\right)^2$ describes the expected loss in profits for the firm at time $t + k$ due to the fact that it will not be able to set a frictionless optimal price that period.
- The summation $\sum_{k=0}^{\infty}$ shows that the firm considers the implications of the price set today for all possible future periods.
- $\beta < 1$ implies that the firm places less weight on future losses than on today's losses.
- Future losses are actually discounted at rate $(\theta\beta)^k$, not just β^k. This is because the firm only considers the **expected** future losses from the price being fixed at z_t. The chance that the price will be fixed until $t + k$ is θ^k. So the period $t + k$ loss is weighted by this probability.

The Optimal Reset Price

- What is the optimal price to set? Differentiate $L(z_t)$ with respect to z_t and set equal to zero.

$$L'(z_t) = 2\sum_{k=0}^{\infty} (\theta\beta)^k E_t \left(z_t - p^*_{t+k}\right) = 0$$

- Separating out the z_t terms from the p^*_{t+k} terms, this implies

$$\left[\sum_{k=0}^{\infty} (\theta\beta)^k\right] z_t = \sum_{k=0}^{\infty} (\theta\beta)^k E_t p^*_{t+k}$$

- Now, we can use our old pal the geometric sum formula to simplify the left side of this equation. In other words, we use the fact that

$$\sum_{k=0}^{\infty} (\theta\beta)^k = \frac{1}{1-\theta\beta}$$

The Optimal Reset Price

- To give us

$$\frac{z_t}{1-\theta\beta} = \sum_{k=0}^{\infty} (\theta\beta)^k \, E_t p^*_{t+k}$$

- This implies a solution of the form

$$z_t = (1-\theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k \, E_t p^*_{t+k}$$

- Stated in English, all this equation says is that the optimal solution is for the firm to set its price equal to a weighted average of the prices that it would have expected to set in the future if there weren't any price rigidities. Unable to change price each period, the firm chooses to try to keep close "on average" to the right price.

Incorporating the Frictionless Price

- And what is this "frictionless optimal" price, p_t^*?
- Assume that the firm's optimal pricing strategy without frictions would involve setting prices as a fixed markup over marginal cost:

$$p_t^* = \mu + mc_t$$

- Thus, the optimal reset price can be written as

$$z_t = (1 - \theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k E_t (\mu + mc_{t+k})$$

- Which simplifies to

$$z_t = \mu + (1 - \theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k E_t mc_{t+k}$$

Inflation Dynamics in the Calvo Model

- It turns out that this type of pricing implies a type of Phillips curve.
- Let's re-examine the optimal reset price equation.

$$z_t = (1 - \theta\beta) \sum_{k=0}^{\infty} (\theta\beta)^k E_t (\mu + mc_{t+k})$$

- We have shown that the first-order stochastic difference equation

$$y_t = a x_t + b E_t y_{t+1}$$

can be solved to give

$$y_t = a \sum_{k=0}^{\infty} b^k E_t x_{t+k}$$

- We can see that z_t must obey a first-order stochastic difference equation with $y_t = z_t$, $x_t = \mu + mc_t$, $a = 1 - \theta\beta$ and $b = \theta\beta$.
- In other words, we can write the reset price as

$$z_t = \theta\beta E_t z_{t+1} + (1 - \theta\beta)(\mu + mc_t)$$

The New-Keynesian Phillips Curve

- The aggregate price level in this economy is just a weighted average of last period's aggregate price level and the new reset price, where the weight is determined by θ:

$$p_t = \theta p_{t-1} + (1 - \theta) z_t,$$

- This can be re-arranged to express the reset price as a function of the current and past aggregate price levels

$$z_t = \frac{1}{1-\theta} (p_t - \theta p_{t-1})$$

- Substituting in the expression for z_t from previous slide.

$$\frac{1}{1-\theta} (p_t - \theta p_{t-1}) = \frac{\theta \beta}{1-\theta} (E_t p_{t+1} - \theta p_t) + (1 - \theta \beta) (\mu + mc_t)$$

- After a bunch of re-arrangements, this equation can be shown to imply

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta} (\mu + mc_t - p_t)$$

  where $\pi_t = p_t - p_{t-1}$ is the inflation rate.

- This equation is known as the **New-Keynesian Phillips Curve**

Inflation in the New-Keynesian Phillips Curve

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}(\mu + mc_t - p_t)$$

- The New-Keynesian Phillips Curve states that inflation is a function of two factors:
  - Next period's expected inflation rate, $E_t \pi_{t+1}$.
  - The gap between the frictionless optimal price level $\mu + mc_t$ and the current price level p_t. Another way to state this is that inflation depends positively on **real marginal cost**, $mc_t - p_t$.
- Why is real marginal cost a driving variable for inflation? Firms in the Calvo model would like to keep their price as a fixed markup over marginal cost. If the ratio of marginal cost to price is getting high (i.e. if $mc_t - p_t$ is high) then this will spark inflationary pressures because those firms that are re-setting prices will, on average, be raising them.

The NKPC, Real Marginal Cost and Output

- The model views inflation as depending on $\mu + mc_t - p_t$, i.e. the deviation of real marginal cost from its frictionless level. But we don't observe data on real marginal cost.
- Still, it seems likely marginal costs are procyclical, and more so than prices. Example: Overtime wage premia: Marginal cost of labour jumps once output levels are high enough to require more than the standard workweek.
- Some implement the NKPC using a measure of the **output gap** (the deviation of output from its potential level) as a proxy for real marginal cost. Denoting the output gap as $\tilde{y}_t$, assume

$$\mu + mc_t - p_t = \lambda \tilde{y}_t$$

- And the NKPC becomes

$$\pi_t = \beta E_t \pi_{t+1} + \gamma y_t$$

where

$$\gamma = \frac{\lambda (1-\theta)(1-\theta\beta)}{\theta}$$

- This approach can be implemented empirically using various measures for estimating potential output.

The "Asset-Price-Like" Behaviour of NKPC Inflation

- The NKPC may look plausible but remember that, combined with rational expectations, it has very strong predictions.
- The equation below is a first-order stochastic difference equation.

$$\pi_t = \beta E_t \pi_{t+1} + \gamma \tilde{y}_t$$

- Thus, we can apply the repeated substitution method to arrive at

$$\pi_t = \gamma \sum_{k=0}^{\infty} \beta^k E_t \tilde{y}_{t+k}$$

- Inflation today depends on the whole sequence of expected future output gaps.
- Thus, the NKPC sees inflation as behaving according to the classic "asset-price" logic that we saw with the dividend-discount stock price model: Past values of all variables, including inflation itself, don't matter, only the present and expectations of the future.

The NKPC and the Lucas Critique

- While evidence for original Phillips curve relationship has disappeared, there is evidence for a relationship of the form

$$\pi_t = \pi_{t-1} + \alpha - \beta u_t$$

  where the lagged inflation term likely reflects how past inflation affects people's expectations

- This relationship is often used and spawned the well-known idea of the NAIRU (the non-accelerating inflation rate of unemployment) defined implicitly by

$$\alpha - \beta u^* = 0 \Rightarrow u^* = \frac{\alpha}{\beta}$$

- If the true model is the NKPC, then the backward-looking Phillips curve might have a good statistical fit because π_{t-1} is likely to be correlated with $E_t \pi_{t+1}$.
- However, NKPC advocates think policy-makers should not rely on this relationship, because changes in policy may produce a break the correlation between $E_t \pi_{t+1}$ and π_{t-1} and at this point the statistical Phillips curve will break down.

The NKPC and Disinflation

- How should a central bank act to reduce inflation? Traditional thinking on this has been heavily influenced by Phillips curves of the form

$$\pi_t = \pi_{t-1} + \alpha - \beta u_t$$

- Because inflation depends on its own lagged values in this formulation means then it would be very difficult to reduce inflation quickly without a significant increase in unemployment. So gradualist policies are the best way to reduce inflation.
- Implications of the NKPC are completely different: Low inflation can be achieved immediately by the central bank announcing (and the public believing) that it is committing itself to eliminating positive output gaps in the future.
- There has been plenty of evidence that reductions in inflation do tend to be costly in terms of lost output and high unemployment.
- Is this because the NKPC is wrong or because governments failed to credibly convince the public of their commitment to lower inflation rates?

A Hybrid New Keynesian Phillips Curve

- We have discussed the New Keynesian Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + \nu x_t,$$

where x_t is a measure of inflationary pressures.

- Many empirical studies have suggested that this formulation has difficulty in explaining the persistence observed in the inflation data.
- Some have proposed a "hybrid" variant:

$$\pi_t = \gamma_f E_t \pi_{t+1} + \gamma_b \pi_{t-1} + \kappa x_t$$

with the lagged element coming from some fraction of the population being non-rational price-setters who rely on past inflation for their current behaviour.

- The solution for this model takes the form

$$\pi_t = \lambda \pi_{t-1} + \frac{\kappa}{1 - \gamma_f \lambda} \sum_{k=0}^{\infty} \left(\frac{\gamma_f}{1 - \gamma_f \lambda} \right)^k E_t x_{t+k}$$

where λ is a solution to

$$\gamma_f \lambda^2 - \lambda + \gamma_b = 0$$

Example: A Hybrid New Keynesian Phillips Curve

- In general, there will be two possible values of λ to solve the so-called characteristic equation of the model. Usually, only one of these values will work as the λ in this formulation.
- Consider the case where the model is

$$\pi_t = \theta E_t \pi_{t+1} + (1-\theta)\pi_{t-1} + \kappa x_t$$

  In this case, the possible solutions of the characteristic equation are $\lambda_1 = 1$ and $\lambda_2 = \frac{1-\theta}{\theta}$.
- If $0 < \theta \leq 0.5$, then the stable solution is

$$\pi_t = \pi_{t-1} + \frac{\kappa}{1-\theta} \sum_{k=0}^{\infty} \left(\frac{\theta}{1-\theta} \right)^k E_t x_{t+k}$$

- Alternatively if $0.5 \leq \theta < 1$, then the stable solution is

$$\pi_t = \left(\frac{1-\theta}{\theta} \right) \pi_{t-1} + \frac{\kappa}{\theta} \sum_{k=0}^{\infty} E_t x_{t+k}$$

Part III

Monetary Policy in The New-Keynesian Model

The Three-Equation New-Keynesian Model

- Most New Keynesian macro takes as its starting point a three equation model.
  1. New Keynesian Phillips curve
  $$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t$$
  where x_t is the gap between output and its long-run potential level and u_t is a random shock.
  2. An Euler equation for the output gap
  $$x_t = E_t x_{t+1} - \sigma \left(i_t - E_t \pi_{t+1} - r_t^n \right)$$
  where r_t^n is a time varying "natural real rate of interest" that is a function of the growth rate of potential output.
  3. And an equation describing how interest rate policy is set, usually described as an explicit interest rate rule.
- We now move on to looking at what form this interest rate rule might take.

The Joint Behaviour of Inflation and Output

- Before discussing monetary policy rules, let's have a quick examination of the joint dynamics of output and inflation in this model.
- The output equation is

$$x_t = E_t x_{t+1} - \sigma \left(i_t - E_t \pi_{t+1} - r_t^n \right)$$

- The inflation equation is

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t$$

- This can be re-written as

$$\pi_t = \beta E_t \pi_{t+1} + \kappa E_t x_{t+1} - \kappa \sigma \left(i_t - E_t \pi_{t+1} - r_t^n \right) + u_t$$

- We can gather together the inflation and output equations in vector form to write the NK model as

$$\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & \sigma \\ \kappa & \beta + \kappa \sigma \end{pmatrix} \begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} \sigma \left(r_t^n - i_t \right) \\ \kappa \sigma \left(r_t^n - i_t \right) + u_t \end{pmatrix}$$

Eigenvalues of A

- Recall from earlier that for models of the form $Z_t = AE_t Z_{t+1} + BV_t$ to have a unique stable solution, we needed all the eigenvalues of A to be less than one.
- In this case, we have

$$A = \begin{pmatrix} 1 & \sigma \\ \kappa & \beta + \kappa\sigma \end{pmatrix}$$

- The eigenvalues satisfy

$$P(\lambda) = (1 - \lambda)(\beta + \kappa\sigma - \lambda) - \kappa\sigma = 0$$

- This can be re-arranged to read

$$P(\lambda) = \lambda^2 - (1 + \beta + \kappa\sigma)\lambda + \beta = 0$$

- $P(\lambda)$ is a U-shaped polynomial. We can show that $P(0) = \beta > 0$, $P(1) = -\kappa\sigma < 0$ and that $P(\lambda)$ greater than zero again as λ rises above one.
- Together, this means one eigenvalue is between zero and one and the other is greater than one.

No Unique Stable Solution

- This seems like a pretty serious problem for the model: In general, there is no unique stable solution. The model turns out to have multiple equilibria and there is nothing to determine which of the equilibria gets chosen.
- How to deal with this? One way is to accept that there are multiple equilibria and to analyse the impact of interest rate changes on output and inflation across a range of different possible equilibria.
- Another approach is to specify that monetary policy follows a particular rule and that the rule is designed to produce a unique stable equilibrium. This is the approach taken in the conventional New Keynesian literature.
- Other approaches exist—for example the research on the "fiscal theory of the price level" uses an equation for the valuation of the government's debt to pin down the behaviour of prices. I don't think these models are relevant enough to the real world to teach, so we are not going to cover them.

A Taylor-Type Rule

- What might a good monetary policy look like?
- Let's start with a rule similar to the one proposed by John Taylor and which has received a huge amount of attention in the monetary policy literature:

$$i_t = r_t^n + \phi_\pi \pi_t + \phi_x x_t$$

- Monetary policy "leans against" inflation and output gaps by raising the interest rate when these increase.
- "Similar" rather than identical because we are allowing the interest rate to move with the natural rate, whereas Taylor's rule has a constant intercept.
- Output equation becomes

$$x_t = E_t x_{t+1} + \sigma E_t \pi_{t+1} - \sigma \phi_\pi \pi_t - \sigma \phi_x x_t$$

- This can be combined with the NKPC to produce a system of first-order stochastic difference equations.

Dynamics under a Taylor Rule

- Let $Z_t = \begin{pmatrix} x_t \\ \pi_t \end{pmatrix}$ and $V_t = \begin{pmatrix} 0 \\ u_t \end{pmatrix}$
- Under this Taylor rule, the economy can be described by a system of the form

$$Z_t = AE_t Z_{t+1} + BV_t$$

where

$$\begin{aligned} A &= \frac{1}{1+\sigma\phi_x+\kappa\sigma\phi_\pi}\begin{pmatrix} 1 & \sigma\left(1-\beta\phi_\pi\right) \\ \kappa & \beta+\sigma\kappa+\beta\left(1+\sigma\phi_x\right) \end{pmatrix} \\ B &= \frac{1}{1+\sigma\phi_x+\kappa\sigma\phi_\pi}\begin{pmatrix} 1 & -\sigma\phi_\pi \\ \kappa & 1+\sigma\phi_x \end{pmatrix} \end{aligned}$$

- This system is a matrix version of the first-order stochastic difference equations and, under certain conditions, it can be solved in a similar fashion to give

$$Z_t = \sum_{k=0}^{\infty} A^k B E_t V_{t+k}$$

Uniqueness and Stability Conditions

- For the model to have a unique stable equilibrium, we need both of the eigenvalues of A to be less than one in absolute value.
- I won't go through calculating the eigenvalues of the A matrix.
- However, it can be shown that both eigenvalues of A are inside unit circle if

$$\phi_\pi + \frac{(1-\beta)\,\phi_x}{\kappa} > 1$$

- Provided the policy rule satisifies this requirement, we get a unique stable equilibrium.

The Taylor Principle

- Interpretation of stability condition:

$$\phi_\pi + \frac{(1-\beta)\,\phi_x}{\kappa} > 1$$

- Quick interpretation: $\beta \approx 1$, so the condition is approximately $\phi_\pi > 1$.
- Nominal interest rates must rise by more than inflation, so real rates rise in response to an increase in inflation.
- Advocated by John Taylor: Now known as the Taylor Principle.
- Why is this needed for stability? Otherwise, inflationary shocks reduce real interest rates, stimulates the economy, and this further stimulates inflation.
- Full interpretation. NKPC implies that in the long-run

$$x_t = \frac{(1-\beta)}{\kappa}\pi_t$$

- Long-run response to inflationary shock

$$\Delta i = \phi_\pi \Delta\pi + \phi_x \Delta x = \left(\phi_\pi + \frac{(1-\beta)\,\phi_x}{\kappa}\right)\Delta\pi$$

Evidence on Monetary Policy Rules

- Clarida, Gali and Gertler (QJE, 2000) and others have argued that the Fed's monetary policy violated the Taylor Principle during the period prior to the appointment of Paul Volcker.
- Estimates from the Volcker-Greenspan era show estimates of θ_π well in excess of one.
- Thus, it has been argued that during the 1960s and 1970s, the Fed was not pursuing stabilizing monetary policy.
- This lack of stabilization may have contributed to macroeconomic instability and the Great Inflation.
- Some arguments about this: Former Fed economist, Athanasios Orphanides argued that if one uses real time data and real time estimates of the output gap, then the Fed thought it was pursuing a policy consistent with $\phi_\pi > 1$.

Part IV

Optimal Monetary Policy

Quadratic Loss Function Framework

- How do we think about what is "optimal" for a central bank to do?
- Clearly, central banks don't like inflation. They would also like to keep output on a steady path close to potential.
- For a long time, economists have formulated central banks as behaving in a way that minimizes a "loss function" something like

$$L_t = \frac{1}{2} \sum_{t=0}^{\infty} \beta^t E_t \left(\pi_{t+k}^2 + \lambda x_{t+k}^2 \right)$$

  where, as before x_t is the output gap and λ indicates the weight put on output stabilization relative to inflation stabilization.
- Economists like quadratic loss functions: When you differentiate things to the power of 2, they give you equations with things to the power of one, i.e. linear relationships.
- Traditionally, though, the quadratic loss function was purely ad hoc.

Woodford's Rationale for the Quadratic Loss Function

- Michael Woodford has shown that one can use the formula

$$L_t = \frac{1}{2}\sum_{t=0}^{\infty} \beta^t E_t \left(\pi_{t+k}^2 + \lambda x_{t+k}^2\right)$$

  as a quadratic approximation to consumer utility in the standard NK model.
- He shows that the correct value is $\lambda = \frac{\kappa}{\theta}$ (κ is coefficient on output gap in NKPC, θ is elasticity of demand for firms.)
- Rationale for the two terms:
  1. x_t^2 term: Risk-averse consumers prefer smooth consumption paths. Keeping output close to its natural rate achieves this.
  2. π_t^2 term: Consumers don't just care about the level of consumption but also its allocation. With inflation, sticky prices implies different prices for the symmetric goods and thus different consumption levels. Optimality requires equal consumption of all items in the bundle. Rationale for welfare effect of inflation, independent of its effect on output (though perhaps you can think of other, better, explanations for a negative effect of inflation on welfare.)

Optimal Policy Under Commitment: Solution

- Suppose that the central bank could commit today to a (state-contingent) strategy that it can adopt now and in the future.
- Lagrangian is

$$\mathcal{L} = \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_t\left[\pi_{t+k}^2 + \lambda x_{t+k}^2 + 2\mu_{t+k}\left(\pi_{t+k} - \beta\pi_{t+k+1} - \kappa x_{t+k}\right)\right]$$

- First-order conditions:

$$\begin{aligned} \lambda E_t x_{t+k} - \kappa E_t \mu_{t+k} &= 0 \\ E_t\pi_{t+k} + E_t\mu_t - E_t\mu_{t+k-1} &= 0 \end{aligned}$$

  for $t = 0, 1, 2, ...$ where $\mu_{-1} = 0$ (The problem does not contain a time $t = -1$ constraint).
- We have $E_t x_{t+k} = \frac{\kappa}{\lambda} E_t\mu_{t+k} = \theta E_t \mu_{t+k}$.
- We also have $E_t\pi_{t+k} = E_t\mu_{t+k-1} - E_t\mu_{t+k} = -\frac{1}{\theta}E_t\Delta x_{t+k} \Rightarrow \Delta E_t x_{t+k} = -\theta E_t\pi_{t+k}$.

Optimal Policy Under Commitment: Characterization

- This means optimal policy will be characterized by

$$\begin{aligned} x_t &= -\theta \pi_t = \theta \left(p_{t-1} - p_t \right) \\ E_t \Delta x_{t+1} &= -\theta E_t \pi_{t+k} = \theta \left(p_{t+k-1} - p_{t+k} \right) \end{aligned}$$

- So, given some initial price level p_{-1}, we get

$$E_t x_{t+k} = \theta \left(p_{-1} - E_t p_{t+k} \right)$$

  because $\pi_t = p_t - p_{t-1}$.
- Optimal policy is set to "lean against the price level."
- Shocks temporarily affect the price level but have no cumulative effect. On average, inflation is zero.
- Note that this policy is history dependent: Policy today depends on the whole past sequence of shocks that have determined today's price level, not just today's shocks.

Optimal Policy Under Discretion

- Suppose that a central bank cannot commit to taking a particular course of action in the future. Instead, all they can do is adopt the optimal strategy for what to do today, and then tomorrow adopt the optimal strategy for what to do tomorrow when it arrives, and so on.
- What difference does this make?
- Recall that the optimality conditions for periods t and $t+1$ were

$$\begin{aligned} x_t &= -\theta \pi_t \\ E_t x_t - E_t x_{t+1} &= -\theta \pi_{t+1} \end{aligned}$$

- So the conditions for the first period are different from the rest. At time t, the previous period, time $t-1$, is gone and doesn't matter now. But we do take into account the effect that time t decisions have at time $t+1$.
- With discretion, the policy makers wake up every day and solve the optimal problem again with all the time subscripts pushed forward. So at time $t+1$ the optimal policy for x_{t+1} is the same as the optimal policy for previously implemented for x_t in the problem we have solved.
- So, under discretion, the policy-maker always sets $x_t = -\theta \pi_t$.

Inflation Under Optimal Discretionary Policy

- Policy implies "leaning against inflation": $x_t = -\theta\pi_t$.
- Inflation can be characterized as

$$\pi_t = \beta E_t \pi_{t+1} - \kappa\theta\pi_t + u_t$$

- New first-order difference equation

$$\pi_t = \left(\frac{1}{1+\theta\kappa}\right)\left(\beta E_t \pi_{t+1} + u_t\right)$$

- Repeated iteration solution:

$$\pi_t = \left(\frac{1}{1+\theta\kappa}\right)\sum_{k=0}^{\infty}\left(\frac{\beta}{1+\theta\kappa}\right)^k E_t u_{t+k}$$

Optimal Policy Under Discretion: AR(1) Shocks

- Often assumed that cost-push shocks are AR(1):

$$u_t = \rho u_{t-1} + v_t$$

where v_t are iid with mean zero.

- This implies that $E_t u_{t+k} = \rho^k u_t$.
- Inflation now becomes

$$\pi_t = \left(\frac{1}{1+\theta\kappa}\right)\left[\sum_{k=0}^{\infty}\left(\frac{\beta\rho}{1+\theta\kappa}\right)^k\right] u_t$$

- Use $\sum_{k=0}^{\infty} c^k = \frac{1}{1-c}$ for $|c| < 1$ to give

$$\pi_t = \left(\frac{1}{1+\theta\kappa}\right)\left(\frac{1}{1-\frac{\beta\rho}{1+\theta\kappa}}\right) u_t = \frac{u_t}{1+\theta\kappa-\beta\rho}$$

Optimal Policy Under Discretion: Interest Rate Rule

- AR(1) cost-push shock thus also implies that $E_t x_{t+1} = \rho x_t$ and $E_t \pi_{t+1} = \rho \pi_t$.
- Can substitute these and $x_t = -\theta \pi_t$ into the Euler equation

$$x_t = E_t x_{t+1} - \sigma \left(i_t - E_t \pi_{t+1} - r_t^n \right)$$

  to back out what the optimal interest rate rule looks like.
- Get a rule of the form

$$i_t = r_t^n + \left(\rho + \frac{(1-\rho)\,\theta}{\sigma} \right) \pi_t$$

- Will be greater than one if $\frac{\theta}{\sigma} > 1$ which will hold for all reasonable parameterizations. Satisfies Taylor Principle.
- Note that inflation and thus interest rates do not depend at all on what happened in the past.

Comparing Policy Under Committment and Discretion

- It can show that committment policy produces a superior welfare outcome to discretionary policy.
- Woodford (2003): "Optimal policy is history dependent ... because the *anticipation* by the private sector that future policy will be different as a result of conditions at date t—even if those conditions no longer matter for the set of possible paths for the target variables at the later date—can improve stabilization outcomes at date t.
- About a transitory cost-push shock u_t: "If the transitory disturbance is expected to have no effect on the conduct of policy in later periods ... then the short-run trade-off between inflation and the output gap at period t is shifted vertically by u_t, requiring the central bank to choose between an increase in inflation, a negative output gap, or some of each. If instead, the central bank is expected to pursue a tighter policy in period $t+1$ and later ... then the short run tradeoff is shifted by the total change in $u_t + E_t\pi_{t+1}$, which is smaller. Hence greater stabilization is possible."
- But there may be problems with implementing this policy and sticking to it.